Vol. 1

# The Great Architects of Mathematics

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ARYA BOOK DEPOT

# THE GREAT ARCHITECTS of MATHEMATICS

## (Volume – I)

## [INDIAN MATHEMATICIANS]

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***Dedicated to***

Prof. (Dr.) B.D. Pant, M.Sc. Ph.D. (Maths)

My most respected Gurujee

who taught me Mathematics

at higher level

# Contents

# 1. *Baudhayana*

Baudhayana was a Vedic Scholar of Yajurvedins Taittriya school, who authored the oldest Sulbasutras. Considerin the view of George Buhler it is believed that Budhayana live probably eight centuries before the Christian era and his nati place was somewhere in Andhra country of South India. Bu Ramgopal had shown different opinion about it by stating that th Srautasutras and Dharmasutras of Baudhayana give sufficie evidence to establish his familiarity with the region between th Ganges and the Yamuna, generally known as Aryavarta. So th region should be his birth place. In truest sense, Baudhayana wa not a mathematician. He was basically a priest and a teacher religious matters. He beautifully formulated and applied th mathematical rules for construction of different altars, vedic yagnakundas which were used for various religious activitie Because of the exactness of these rules and their geometrica significance the Sulbasutras of Baudhayana are considered th finest inventions of ancient Hindu scholars. The Sulbasutras Baudhayana are spread in twenty one chapters, the detail of topic covered in these chapters is given below for the knowledge of th readers.

- **Chapter 1 :** Units of measurements, construction of square and rectangles, theorem of square on the diagonal and surds
- **Chapter 2 :** Transformation of geometrical figures.
- **Chapter 3 :** Sacrificial fires and Altars.
- **Chapter 4 :** Areas of Pragvamsa (rectangle), Mahavedi (Isosceles trapezium) etc. Construction of Ekadasi and Asvamedha vedi and the value of $\pi$.
- **Chapters 5, 6 and 7 :** Enlargement of firealtars.
- **Chapters 8 and 9 :** Construction of a Rectilinear Syenacit (falcon shaped firealtars) of first and second type.

- **Chapters 10 and 11 :** Construction of a fire altar in the form of a falcon with curved wings and extended tail first and second type.
- **Chapters 12 and 13 :** Construction of a fire altar in the form of a Kankacit (Kite) and an Alaja bird.
- **Chapters 14, 15, 16 and 17 :** Construction of fire altar in the form of Praugaciti (Isosceles triangle), Ubhayata Prauga (Rhombus), Rathacakraciti (Chariot wheel) and a square trough.
- **Chapter 18 :** Construction of fire altars in the form of a circular trough and of Samuhya and Paricayya.
- **Chapter 19 :** Construction of fire altars in the form of Samasanacit (Pyre).
- **Chapters 20 and 21 :** Construction of firealters in the form of a Kurmacit (Tortoise) with twisted and rounded limbs.

Now we explain some of the important results given by Baudhayana. First we take a very important result related to the sides of any right angled triangle which is expressed in this sloka as :

दीर्घस्याक्ष्णया रज्जुः पार्श्वमानी तिर्यङ्‌मानी
च यत्पृथग्भूते कुरुतस्तदु भयं करोति ॥

(The diagonal of rectangle produces both areas which its length and breadth produce separately)

For any rectangle PQRS, this sutra states that $PR^2 = PQ^2 + QR^2$. This proposition was given first by Baudhayana (800 B.C.), which later on was also expressed in almost same language by other Sulbakaras Apastamba (A.Sl. 1.4), Katyayana (K. Sl. 2.7) and Manava (M.Sl. 10.10). Baudhayana also made it further clear that this theorem can be easily verified from the relations like

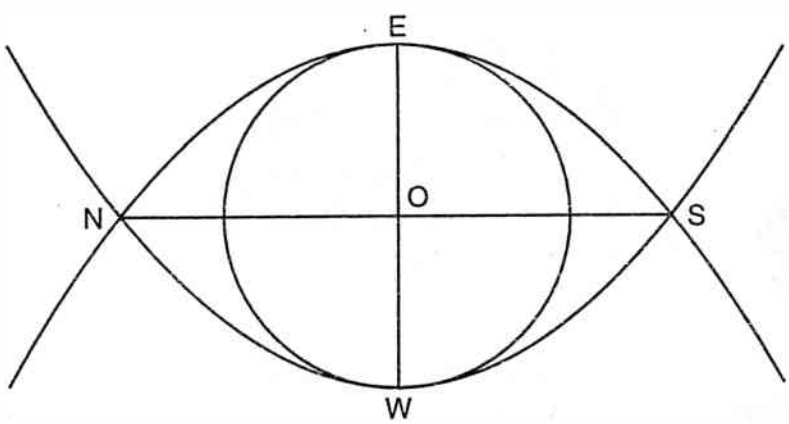
$$3^2 + 4^2 = 5^2$$
$$12^2 + 5^2 = 13^2$$
$$15^2 + 8^2 = 17^2$$
$$7^2 + 24^2 = 25^2$$
$$12^2 + 35^2 = 37^2$$
$$15^2 + 36^2 = 39^2$$

Though the proof of this theorem given by Baudhayana and other Sulbakars could not be traced but the scholars like Burk, Hankel, Thaibout and Dalta conjectured that Baudhayana was well aware of the proof of this theorem and the proof of Baudhayana

may be any one of the proofs which are known presently as the proofs given by ancient Hindu mathematicians to obtain such type of results. Unfortunately this theorem is attributed to the Greek philosopher Pythagoras (C.540 B.C.) and known as Pythagoras theorem universally but it will be proper if this theorem is named as Baudhayana theorem or Sulba theorem because there is no doubt in accepting the fact that Baudhayana established this relationship even two to three centuries before the period of Pythagoras. Moreover, it will also be appropriate to mention here that ancient Hindus were aware of such relationship even two thousand years before the christian era, where we see a clear reference of the relationship like $39^2 = 36^2 + 15^2$ in the Taittriya Samhita (before 2000 B.C.) and also in Satapatha Brahmana (X. 2.3.7).

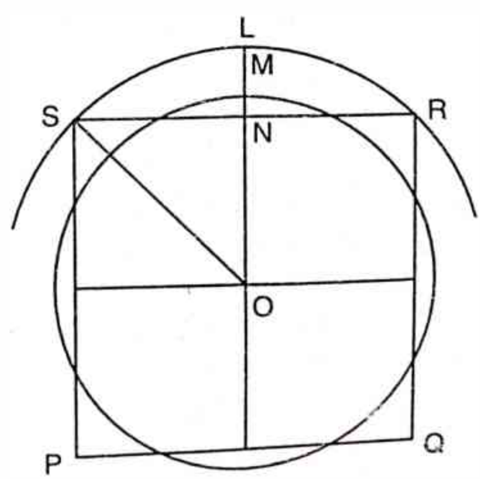
The second important result of Baudhayana is given below :

(*ii*) This result is also known as fish method to draw two perpendicular diameters in a circle.




The third important result of Baudhayana is given below :

(*iii*) Here we give one of the classic problems of geometry *i.e.* circling the square which is beautifully explained by Baudhayana in his rule that "To circle a square, take half of its diagonal towards east-west line and then describe a circle keeping third part of it outside the square" : This rule is demonstrated geometrically in adjoining figure.




For given square PQRS, get central point O, join OS and draw arc SLR, divide LN in such a way that

LM = 2MN draw a circle with radius OM, which is approximately equal in area that of the area of square PQRS.

From this construction the approximate value of $\pi$ can also be obtained. If $a$ is the side of square PQRS, then

$$OS = \frac{a}{\sqrt{2}} \text{ and } OM = \frac{a}{6}(\sqrt{2}+2)$$

or $$2\,OM = \frac{a}{3}(\sqrt{2}+2)$$

Let $$2\,OM = d,$$

$\therefore$ $$d = \frac{a}{3}(\sqrt{2}+2) \quad \text{or} \quad \frac{9d^2}{(2+\sqrt{2})^2} = a^2$$

but $$a^2 = \frac{\pi d^2}{4} \qquad \text{so} \quad \pi = \frac{36}{(2+\sqrt{2})^2}$$

or $$\pi = \frac{36(2-\sqrt{2})^2}{4} = 9(2-\sqrt{2})^2$$

or $$\pi = 9(6-4\sqrt{2})$$

Putting $\sqrt{2} = 1.414...$ approximately

$$\pi = 3.088 \text{ (approximately)}$$

Baudhayana also described the value of $\pi$ in his sutra as

$$\pi = 4\left(1 - \frac{1}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}\right)$$
$$= 3.0885.$$

Baudhayana considered several values of $\pi$ while constructing different circular figures. In such figures he considered approximate values of $\pi$ like $\frac{676}{225} \sim 3.004$, $\frac{990}{289} \sim 3.114$ and $\frac{1156}{361} \sim 3.202$. Though these values of $\pi$ were not accurate but in the construction of different altars these approximate values of $\pi$ could not lead the result to any significant error, that is why, Baudhayana took these values as and when they were required for the construction of geometrical figures of his choice. Baudhayana also devised the rules to find the solutions of general linear equations of one variable and also dealt the problems related to indeterminate equations of first degree. In his Sulbasutras we also find beautiful exposition of geometrical techniques to solve various type of algebraic equations like $ax^2 = c$ and $ax^2 + bx = c$. Baudhayana also recognised the importance of numbers of incommensurable nature and calculated their approximate value. He used these numbers in the process of construction of various geometrical figures.

# 2. *Manava*

Manava was the Vedic Scholar of the Maitrayani School. Abou the place of origin of Manava, nothing can be said with full certainit because of nonavailability of authentic informations related to hi birth and place of birth in the literature of that period. But man historians expressed that Sulbasutras of Manava were compose around 750 B.C which sufficiently placed him in between the perio of Baudhayana (800 B.C.) and Apastamba (600 B.C.). Manava' Sulbasutras represent the Brahmana geometry and mensuratio and he wrote those sutras almost in archaic style. The genera impression of the historians about all Sulbakaras also applies t Manava that he was not a mathematician in true sense but he wa a man of considerable learning who had formulated various rule and expressed them beautifully to define and construct differen geometrical figures. He used them for various religious purposes It also become clear from the available literature of that time tha Manava was a Vedic Priest as well as a skilled craftsman. Manava Sulbasutra followed the common tradition of other Sulbakars and continued the same during his period to pass that valuable treasure of mathematical knowledge from one generation to other. Though at several places it is also observed by the scholars that Sutras expressing the details and methods related to various geometrica figures given by Manava are difficult to comprehend at the first instance and these are understood only by taking into considerations the references of sutras of other Sulbakaras like Baudhayana Apastamba and Katyayana. The Sulbasutras of Manava are detailed in sixteen chapters and the topics covered in these chapters are given below :

- **Chapter 1 :** Determination of east-west line, construction of Darsiki Vedi, size and places of Garhapatya, Ahavaniya, Daksinaghi, Utkara and rule for construction of a square.
- **Chapter 2 :** Units of Chariot, Construction of Pasubandha, Pasuki, Maruti, Varuna and Paitriki Vedis.

- **Chapter 3 :** Positions of Pragvamsa, Sadas and Havirdhana, relative to Mahavedi.
- **Chapters 4 and 5 :** Units of measures and weights, bricks, construction of Caturasrasyenacit.
- **Chapter 6 :** Garhapatya, Agnidhriya, Brahmnacchamsa, Marjaliya.
- **Chapters 7 and 8 :** Construction of suparnaciti
- **Chapter 9 :** Areas of Garhapatya, Dhisnyas and placing of bricks in different Yagnas.
- **Chapter 10 :** The Sulbavid, Sanku, Rope measurement of volume, Properties of right angled triangle.
- **Chapter 11 :** Units of measurement, circling a square, areas of plane figures, value of $\pi$, quadrature of the circle, use of Pancangi cord, measures for diagonal of a rectangle.
- **Chapter 12 :** Diagonal of a right triangle.
- **Chapter 13 :** Construction of Saumiki Vedi, Garhapatya, Caturasrasyena of another type, Agnidhriya, Hotriya, Brahmanacchamsa and Marjaliya.
- **Chapter 14 :** Vakrapaksa Syena, Kanka and Alaja.
- **Chapters 15 and 16 :** Praugacit, Ubhayata Prauga, Samuhya, Drona, Rathacakracit.

Manava gava the rule to construct Darsiki Vedi in the form of isosceles trapezium having base 64 ang., face 48 ang. and altitude 96 ang. The Pasuki Vedi and Maruti Vedi were also constructed in the form of isosceles trapezium. He gave the rule to construct a square by following the relation $a^2 + (3/4\ a)^2 = (5/4\ a)^2$, where $a$ is length of the chord. Manava calculated the value of $\pi$ by following the relation $c = \frac{d}{5} + 3d = 3.2$ or $\frac{c}{d} = 3.2$, where $c$ is circumference of circle and $d$ the diameter. He also described the properties of right angled triangle and established the formula $(3n)^2 + (4n)^2 = (5n)^2$, where $n$ is any quantity. He also expressed the methods to construct rational right angled triangle with sides measured in fractions *i.e.* $2\frac{1}{2}, 6, 6\frac{1}{2}$ ; $7\frac{1}{2}, 10, 12\frac{1}{2}$ in addition to the sides having the values in terms of whole numbers like 72, 96, 120 and 40, 96, 104. Manava also gave the rule to construct Saumiki fire altars in the form of an isosceles trapezium having base $12\sqrt{3}$, face $8\sqrt{3}$ and altitude $12\sqrt{3}$ which is sufficient to understand that Manava was quite comfortable in dealing with the numbers of incommensurable nature.

# 3. Apastamba

Apastamba was a great scholar of Vedic lore, who probably liv
six centuries before the Christian era. He was one of the promine
authors of Sulba-Sutras which are the part of the Srautasutr
belonging to all the four Samhitas. Sulbasutras were basically mea
for the construction of altars/vedis to perform Vedic rituals. T
Sulbasutras of Apastamba are very interesting because of th
geometrical importance. These are attached to Black Yajurvedi
of the Taittriya School. The exact period of Apastamba's life span
also not known but most of the scholars and historians are of t
view that his Sulbasutra appeared after the period of another Ved
Scholar Baudhayana. George Buhler expressed his opinion th
Apastamba lived in South India particularly in Andhra, whe
several land-grants are found, which are also referred in th
Dharmasutra's of Apastamba and probably lived in the fifth-six
century before Christian era. It is also believed from the availab
literature of that period that Apastamba was a priest and taug
the matters related to religious practices. The Sutras formulate
by him were mainly concerned with religious activities to facilita
the construction of the altars of yagnas, bhumikas and sacrifice
Apastamba's Sulbasutras are spread into six sections, twent
chapters covering altogether two hundred and twenty three Sutra
The topics dealt in these chapters are detailed below :

- **Chapter 1 :** Construction of square, the theorem of squa on the diagonal and the value of square root of 2.
- **Chapter 2 :** Construction of square, surd, a square from combination and difference of two squares and transforma tion of a rectangle into a square.
- **Chapter 3 :** Transformation of a square into a rectangle an a circle, a circle into a square and construction as well a enlargement of squares.

- **Chapter 4 :** Garhapatya, Ahavaniya and Daksinagni, their relative positions and distances and construction of Darsikya Vedi.
- **Chapters 5 and 6 :** The method of cords and their use in the construction of altars.
- **Chapter 7 :** Construction of Sadas, Uparavas, Garhapatya, Dhisnya and Angidhriya.
- **Chapters 8 and 9 :** Characterstics of Agni and construction of Vedis.
- **Chapter 10 :** Construction of a rectilinear syenacit.
- **Chapter 11 :** Construction of a rectilinear syenacit with square bricks.
- **Chapter 12 :** Firealtars in the form of isosceles triangle, rhombus and chariot wheel.
- **Chapter 13 :** Construction of firealtars in the form of chariot wheel and a trough.
- **Chapters 15 to 20 :** Construction of a firealtar in the form of a falcon with curved wings and extended tail with first and second type.

In Apastamba's Sutra we also find reference of decimal numeration system, where a number 972 was named as astavimsatyunam sahasram *i.e.* 28 less than 1000. Interestingly enough, Apastamba remarkably covered the numbers of incommensurable nature. He gave rational approximation of irrational number $\sqrt{2}$ correct to five places of decimals as

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}$$

in this sloka,

समस्य द्विकरणी । प्रमाणं तृतीयेन वर्धयेत्
तच्चतुर्थेनात्म चतुसास्त्रिशोनेन सविशेष:

Ap. Sl. I. 5

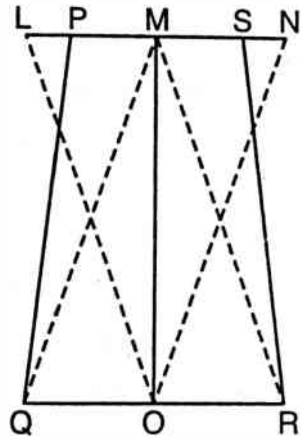
Apastamba was also familiar with such type of other numbers like $\sqrt{3}$ (trikarani) etc. and used them for the construction of various geometrical figures. In his Sutra we find a very useful method for calculation of square root approximately as

$\sqrt{A} = \sqrt{a^2 + r} = a + \frac{r}{2a}$, where $r$ is relatively small *e.g.*

$$\sqrt{17} = \sqrt{4^2 + 1} = 4 + \frac{1}{2 \times 4} = 4.12...$$

which is correct upto two places of decimals.

Apastamba described the methods to construct Mahavedi by using one chord and two chords, which remain in the shape of isosceles trapezium having base 30 prakramas, height 36 prakramas and face 24 prakramas. In the first method one cord is used to construct a right angled triangle and then with the use of this triangle isosceles trapezium is constructed as shown in adjoining figure.



(Mahavedi)

In second method of construction of mahavedi two pieces of chords are used for construction of right angled triangle and then with the help of this triangle isosceles trapezium is constructed. The three sets of chord relations are given by Apastamba in his Sulbasutras as

(*i*) $3^2 + 4^2 = 5^2$

$$(3 + 3\,.\,3)^2 + (4 + 4\,.\,3)^2 = (5 + 5\,.\,3)^2$$

or $$12^2 + 16^2 = 20^2$$

$$(3 + 3\,.\,4)^2 + (4 + 4\,.\,4)^2 = (5 + 5\,.\,4)^2$$

or $$15^2 + 20^2 = 25^2$$

(*ii*) $$5^2 + 12^2 = 13^2$$

$$(5 + 2.5)^2 + (12 + 2\,.\,12)^2 = (13 + 2\,.\,13)^2$$

or $$15^2 + 36^2 = 39^2$$

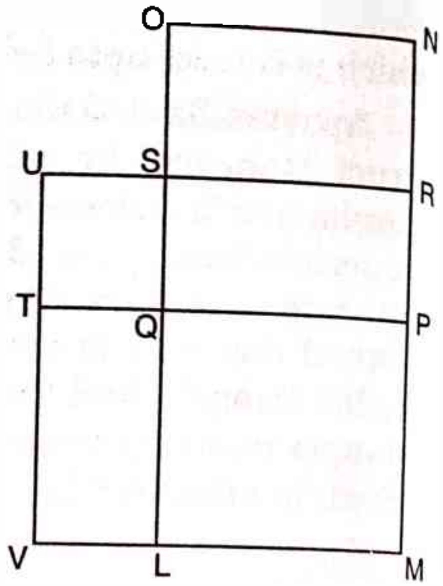
(*iii*) $$8^2 + 15^2 = 17^2$$

$$12^2 + 35^2 = 37^2$$

Apastamba was known about the fact related to rectangle that a square on the diagonal of a rectangle is equal to the sum of the squares on the two adjacent sides. He also described the rule to construct a square equal in area to the given rectangle in this sloka :

दीर्घं चतुरश्रं समचतुरश्रं चिकीर्षन्
तिर्यङ्गं मान्यापच्छिद्य शेषं विभज्योभयत उपदध्यात्
खण्ड यागन्तुना संपूरयेत तस्य निह्रासः उक्तः

By this rule, first lay off the shorter side of given rectangle on the longer side such that QL = LM = MP = QP = ON. Further draw SR bisecting segments OQ and NP and then extend PQ to T, RS to U and ML to V such that LV = QT = SU = QS and join the points U, T and V, which gives required square UVMR.




Apastamba also gave the rule to solve general linear equations. We find some references of indeterminate equations of first degree like $x + y = 21$ and $\frac{x}{a^2} + \frac{y}{b^2} = 1$ in the literature of that period which can be solved by using Apastamba's sutras.

# 4. *Katyayana*

Katyayana was a famous Sulbkara of Shukla Yajurvedin tradition. It is supposed that he lived around 350 B.C. in between the period of Panini and Patanjali. Katyayana gave very systematic and logical treatment to various geometrical rules for the construction of altars and sacrificial fires and gave succint expositions of those rules. Though Katyayana had not composed these sutras solely for mathematical reasons. These sutras were in fact the product of his necessities to perform various religious rites concerned with yagnabhumiks. Among all the Sulbakaras, Katyayana only gave clear exposition of the meaning of Sulba as rule of cord by stating his Sutras from the words "रज्जुसमासं वक्ष्याम:" Katyayana's Sulbasutra's are described in six chapters and the topics covered in these chapters are given below :

- **Chapter 1 :** Drawing of east-west line, construction of squares and fixing the places of the Ahavaniya, Garhapatya, Daksinagni and Utkara altars.
- **Chapter 2 :** Units of measures, Paitrki Vedi, Theorem of square.
- **Chapter 3 :** Difference of two squares, transformation of figures like rectangle into a square and vice versa. Circling a square and area of the circle.
- **Chapter 4 :** Transformation of a triangle in rhombus into a square, construction of Dronacit.
- **Chapters 5 and 6 :** Construction of a square equal to $n$ times a given square and Ekadasini fire altar.

To draw Paitriki Vedi, Katyayana gave a simple rule that joining the midpoints of the sides of a square Paitriki Vedi can be formed. Katyayana also gave a beautiful Sutra to draw squares on the sides and diagonals of a rectangle by dividing them into unit squares and then adding those units together. He expressed it in this Sutra.

यावत्प्रमाणा रज्जुर्भवति तावन्तस्तावन्तो
वर्गा भवन्ति, तान् समस्येत् !!

Katyayana took various combination of numbers including irrational numbers to estabilish the relationship in between the sides of a right angled triangle like

$$1^2 + 3^2 = (\sqrt{10})^2$$

$$2^2 + 6^2 = (\sqrt{40})^2$$

$$1^2 + (\sqrt{2})^2 = (\sqrt{3})^2$$

$$na^2 = \left[\frac{(n+1)a}{2}\right]^2 - \left[\frac{(n-1)a}{2}\right]^2$$ ; where $a$ is rational integer

With the help of this last identity Katyayana gave the rule to construct a square equal to $n$ times the given square.

In Katyayana's Sutra we also find the rule to transform a rectangle into a square which can be expressed in algebraic terms as

$$uv = \left(u - \left(\frac{u-v}{2}\right)\right)^2 - \left(\frac{u-v}{2}\right)^2$$

or $$\left(\sqrt{uv}\right)^2 + \left(\frac{u-v}{2}\right)^2 = \left(\frac{u+v}{2}\right)^2$$

Substituting $r^2$ and $s^2$ for $u$ and $v$ respectively, to eliminate irrational quantities we get,

$$r^2 s^2 + \left(\frac{r^2 - s^2}{2}\right)^2 = \left(\frac{r^2 + s^2}{2}\right)^2$$

which gives the rational solution of the equation of the form

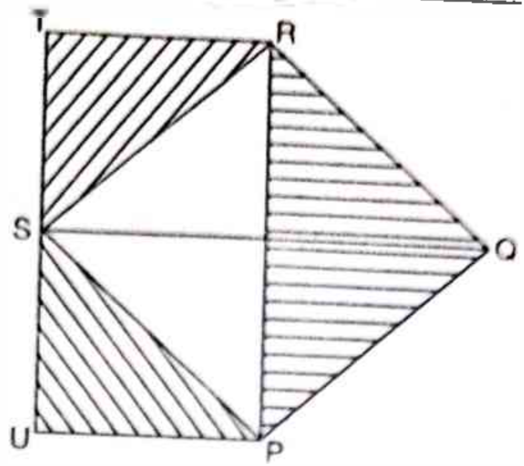
$$x^2 + y^2 = z^2.$$

Katyayana also explained the meaning of dvikarani, trikarani and tritiyakarani in his sutra and described the method to construct sautramaniki vedi, which has the form of isosceles trapezium of base $\frac{30}{\sqrt{3}}$ prakrams, face $\frac{24}{\sqrt{3}}$ prakrams and altitude $\frac{36}{\sqrt{3}}$ prakrams.

To transform a rhombus Katyayana gave the Sutra as

उभयतः प्रउगं चेन्मध्ये तिर्यगापच्छिद्य पूर्ववत् समस्येत् !!

By this sutra rhombus is first divided into two isosceles triangles by joining diagonal and then divide it into four right triangles by




cutting them along their altitudes and when these four right triangles are joined together, they form required rectangle as shown in figure.

It is also interesting here to mention that Katyayana gave the solution of the quadratic equations of the forms $ax^2 + bx = c$ and $ax^2 = c$, while dealing with falcon shaped fire altar. He involved the equations of this nature and solved them by using following steps :

$$7u^2 + \frac{1}{2}u = 7\frac{1}{2} + p$$

which gives, $$u = \frac{1}{28}(\sqrt{841 + 112p} - 1)$$

and after simplifying it and neglecting the higher powers of $p$ Katyayana gave the value

$$u^2 = 1 + \frac{p}{7}.$$

# 5. *Aryabhata I*

Year of Birth 476 A.D.

Year of Death 550 A.D.

Aryabhata I was born in the year 476 A.D ; which is calculated on the basis of the description given by him in this sloka that :

षष्ट्यब्दानां षष्टिर्यदा व्यतीतास्त्रयश्च: युगपादा:
त्र्यधिका विंशतिरब्दास्तदेह मम जन्मनोऽतीता:

(A.B. Kalkriyapad 10)

From this sloka, the year of birth of Aryabhata I is computed in the way as 60 × 60 = 3600 and 3600 – 3179 = 421 saka year. Further 421 + 78 = 499 A.D., it shows that in the year 499 A.D. Aryabhata I was twenty three years old, so his year of birth should be 499 – 23 = 476 A.D. Regarding his place of birth nothing definite can be said but some scholars of later period who had affiliation with Aryabhata school, only claimed that he belonged to Asmaka region. Many other historians had also put forth their own conjectures about him by suggesting that his native place probably may be somewhere in Southern India either in Kerala or in Andhra region, but they could not corroborate these conjectures with any convincing evidences. Due to such differences among the scholars Parameswaran also wrote "... no final verdict can be given regarding the locations of Asmakajanapada and Kusumpura". But one point can be made with full certainity that Aryabhata I came to Kusumpura (Patiliputra or Patna) for acquiring knowledge in the areas of mathematics and astronomy. In Ganitapada, Aryabhata himself expressed that :

आर्यभटस्त्विह निगदति कुसुमपुरेऽभ्यर्चितं ज्ञानम्

Aryabhata I was exceptionally brilliant in his scientific way of thinking which only lead him to discover several new laws, rules and principles related to mathematics and astronomy. In a very young age of twenty three years only, Aryabhata I authored his celebrated text Aryabhatiya. This text was beautifully expressed

in the form of 121 Sanskrit slokas. It was his sheer medha only that the topics ranging from mathematics to astronomy, which were of far reaching importance, he could condense in these 121 slokas, which made this text a precious gift of Aryabhata I in the field of mathematics and astronomy to the posterity. This text consists of four parts *viz.* Dasagitikapad, Ganitapad, Kalkriyapad and Golapad. The first part Dasgitikapad comprises of 13 slokas and covered the topics like large units of time namely Kalpa, Manu and Yuga ; Circular units of arc *i.e.* degree and minute and linear units like yojana, hasta and angula. The positions of the planets, their apogees or aphelia, diameter of the planets, the inclinations of the orbital planes of the planets with the ecliptic and the epicycles of the different planets and construction of sinetables is also explained in this section. The second part of the Aryabhatiya is Ganitapad which consists of 33 slokas and deals with topics like geometrical figures and their properties ; mensuration, problems on the Sanku Chaya (shadow of the gnomon) ; arithmetic and geometric progression (Sredhi) ; simple and compound interest, square root, cube root, rule of three, simultaneous linear and quadratic indeterminate equations ; first order indeterminate equations in two unknowns of the form $ax \pm c = by$, for the integers $a$, $b$, and $c$ etc. The third part of this text *i.e.* Kalakriyapad contains 25 slokas related to reckoning of time explaining various units of time, calcaulation of adhikamesa, Kasyatithis, speeds of planetary motion etc. The fourth part of this text is Golapada which deals with arithmetic and astronomy. It has 50 slokas covering topics related to sphere, the celestial equator, shape of the earth, the cause of day and night, lunar and solar eclipses, etc. Though it is believed by scholars that Aryabhata I wrote only one text Aryabhatiyam during his life time but historian K.S. Shukla found another portion of Aryasiddhanta of thirty four slokas ; On the basis of which he concluded that Aryabhata I might have authored Maha Arya Siddhanta covering the related topics based on midnight astronomy because this concept of Aryabhata I was referred in Mahabhaskariya, Laghubhaskariya and some other texts and commentaries of later period. The first commentary on Aryabhatiya was written by Prabhakar but unfortunately it is not available now. Thereafter Bhaskara I wrote the commentary on this text around 629 A.D. Bhaskara I called Aryabhata with the name Asmaka, his text Aryabhatiya by Asmakatantra and his followers by Asmakiya. Someswara wrote another commentary on Aryabhatiya around 1040 A.D. Besides these, Suryadeva, Nilakantha and Paramesvara also wrote commentaries on it. Few

other commentaries were also written by the scholars in the languages Telugu and Malayalam.

In the second sloka of the Dasgitikapad Aryabhata I expressed his innovative alphabetical numerical system. In which he used thirty three letters of the Indian alphabet. With the help of this alphabetic numeration system the numbers from 1 to $10^{18}$ can be represented.He described this system in following sloka as :

वर्गाक्षराणि वर्गे ऽ वर्गे ऽ वर्गाक्षराणि कात् ङ्मौ यः

खद्विनवके स्वरा नव वर्गे ऽ वर्गेनवान्त्य वर्गे वा:

(Second sloka, Dasgitika, Aryabhatiya)

According to this sloka, the first twenty five consonants are assigned the whole numbers from 1 to 25, the twenty sixth alphabet is assigned the value five more than the twenty fifth and remaining seven alphabets get values increased by ten and last consonant has the value one hundred. Aryabhata I's alphabetic numeration system is made explicitly clear in the following table :

**Varga letters and the numbers**

| | | | | | |
|---|---|---|---|---|---|
| Varga | | | | | |
| Ka–Varga | क् | ख् | ग् | घ् | ङ् |
| | k | kh | ga | gha | ṅa |
| | 1 | 2 | 3 | 4 | 5 |
| Ca–Varga | च् | छ् | ज् | झ् | ञ् |
| | cha | chha | ja | jha | ṅa |
| | 6 | 7 | 8 | 9 | 10 |
| ṭa–Varga | ट् | ठ् | ड् | ढ् | ण् |
| | ta | tha | da | dha | ṅa |
| | 11 | 12 | 13 | 14 | 15 |
| ta–Varga | त् | थ् | द् | ध् | न् |
| | ta | tha | da | dha | na |
| | 16 | 17 | 18 | 19 | 20 |
| pa–Varga | प् | फ् | ब् | भ् | म् |
| | pa | pha | ba | bha | ma |
| | 21 | 22 | 23 | 24 | 25 |

**Avarga letters and the numbers**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| य | र | ल | व | श | ष | स | ह |
| *ẏ* | *r* | *l* | *v* | *ś* | *ṣ* | *s* | *h* |
| 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

**Vowels (Nine numbers)**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| अ | इ | उ | ऋ | लृ | ए | ऐ | ओ | औ |
| *a* | *i* | *u* | *ṛ* | *ḷ* | *l* | *ai* | *au* | *au* |
| 1 | 100 | 10000 | | | | | | |
| $(10^0)$ | $(10^2)$ | $(10^4)$ | $(10^6)$ | $(10^8)$ | $(10^{10})$ | $(10^{12})$ | $(10^{14})$ | $(10^{16})$ |

In this system of numeration the notational positions were also classified as Varga and Avarga, from right to left by denoting units place, hundreds place, ten thousands place etc as Varga place and ten's place, thousands place etc as Avarga place. It also becomes clear with such distinction of notational position that Aryabhata I was familar with decimal system of numeration. In the second sloka of Ganita sloka he expressed that

एकं च दशं च शतं च सहस्रमयुत नियतं तथा प्रयुतम् ।
कोट्यर्बुदं च वृंद स्थानात् स्थानं दश गुणं स्यात् !!

(Second sloka, Ganitapada, Aryabhatiya)

According to it the value of every place can be obtained by taking ten times of the preceding value *i.e.* eka, dasa (ten), sat (hundred), sahasra (hundred), ayuta (ten thousand), niyuta (hundred thousand), prayuta (million), koti (ten million), arbuda (hundred million) and v ṛnda (thousand million).

It is noteworthy to mention here that Aryabhata I devised the methods to extract square roots and cube roots depending on the decimal place value system and process of division. Though Aryabhata I followed his new alphabetic numeration system for most of his calculations related to astronomy and mathematics and it is understood that he was also familiar with numeral system. Ifrah also wrote that "... it is extremely likely that Aryabhata I knew the sign for zero and numerals for the place value system. This supposition is based on the following two factors, first, the invention of his alphabetical counting system would have been impossible without zero or the place value system : secondly, he carries out calculations on square and cube roots which are impossible if the numbers in question are not written according to the place value system and zero".

Aryabhata I gave the solutions of quadratic equations and solved the problems related to interest by using it. For the equation of the form $tx^2 + px - Ap = 0$, where $t$ is time, $p$ is principal sum, $x$ the unknown interest and A is amount, he gave the value of $x$ in the form,

$$x = \sqrt{Apt + \left(\frac{p}{2}\right)^2} - \frac{p/2}{t}$$

He also solved the problems related to linear equations and system of linear equations by using the rule of false position. He explained the concept of progression and presented the solutions to the problems related to arithmetic and geometric progressions. To find the number of terms of an arithmetic progression he gave the formula

$$n = \frac{1}{2}\left\{\frac{\sqrt{8bs + (2a-b)^2} - 2a}{b} + 1\right\}$$

where $a$ is the first term, $b$ common difference, and $s$ sum to $n$ terms of given arithmetic progression. He also formulated following sums :

(*i*) $1 + 2 + 3 + \ldots\ldots + n = \dfrac{n(n+1)}{2}$

(*ii*) $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(*iii*) $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$

Aryabhata I also gave solutions for the simultaneous quadratic equation of the form $x - y = d$ ; $xy = b$ as

$$x = \frac{1}{2}(\sqrt{d^3 + 4b} + d);\ y = \frac{1}{2}(\sqrt{d^2 + 4b} - d)$$

It is interesting to note that Aryabhata I was the first mathematician, who gave a method to find the general solution in positive integers of the simple indeterminate equation $by - ax = c$ for integral values of $a, b, c$ and also indicated the way to get positive integral solutions of simultaneous indeterminate equations of the first degree like

$$\begin{aligned} N &= a_1 x_1 + r_1 = a_2 x_2 + r_2 = a_3 x_3 + r_3 \\ &= \ldots\ldots\ldots\ldots\ldots = a_n x_n + r_n. \end{aligned}$$

These type of the equations were called Kuttaka by ancient Hindu mathematicians which has the meaning breaking or pulver-

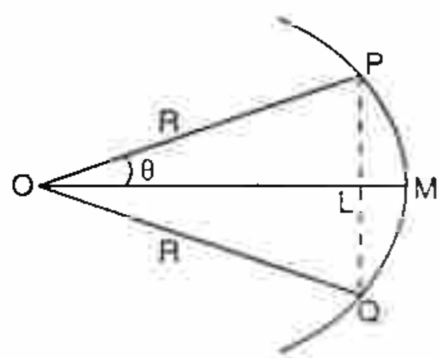
ising. In fact Aryabhata I dealt with these type of equations to solve the problems of following nature :

(*i*) to find a number (N), which when divided by a number $p$ leaves remainder $r_1$ and when divided by another number $q$ leaves remainder $r_2$ *i.e.* $N = px + r_1 = qy + r_2$

(*ii*) to find a number N which leaves remainders $r_1$, $r_2$, $r_3$ ...... $r_n$ respectively when divided by $p_1, p_2, p_3, \ldots, p_n$ respectively

*i.e.* $$N = p_1x_1 + r_1 = p_2 x_2 + r_2 = p_3 x_3 + r_3$$
$$= \ldots\ldots = p_n x_n + r_n.$$

Aryabhata I contributed remarkably in the area of trigonometry, which is named as Jyotpattiganita in Hindu astronomy. The ancient Hindu astronomers generally used three trignometric functions *i.e.* Jya(sine), Kotijya (cosine), and Utkramajya(versine) which are trignometric ratios with reference to an arc of a circle. In the following figure arc $\overset{\frown}{PMQ}$ looks like bow and chord PLQ is chord jya. The half chord PL is referred as ardha-jya. Aryabhata I was the first mathematician who introduced the concept of half chord to define jya function and laid the foundation of modern trigonometry. His basics related to trignometry were different than the old Greek's system. With this concept to half chord, jya of arc PM is PL ; Kotijya of arc PM is OL and arc PM is PL ; Kotijya of arc PM is OL and Utkrama-jya of arc PM is LM, which gives the value of jya $\theta = R \sin \theta$,




Kotijya $\theta = R \cos \theta$ and Utkrama – jya $\theta = OM - OL = R - R \cos \theta$

$$= R(1 - \cos \theta) = \text{vers} \sin (\theta)$$

Aryabhata I constructed the sine table by taking the differences between the successive R sines for arcs of every 3° 45′ *i.e.* 225′ in a circle of radius 3438′. He expressed the rule for the construction of the sine-table in the sloka as :

मखि भखि फखि धखि णखि ञखि
ङखि हस्झ स्ककि किष्ग श्घकि किघ्व
घ्लकि किग्र हक्य धकि किच
स्ग श्झ ङ्व क्ल प्त फ छ कलार्धज्या

According to it, the R sine differences at the intervels of 3° 45′ *i.e.* 225′ of arc are 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22 and 7″.

The sine table of Aryabhata I gives remarkably very close values to the exact values computed in modern notations, considering the radius 3438′.

**Sine table of Aryabhata I**

| *Angle* (θ) | *jya* (θ) | *Angle* (θ) | *jya* (θ) |
|---|---|---|---|
| 3°45′ | 225 | 48°45′ | 2585 |
| 7° 30′ | 449 | 52° 30′ | 2728 |
| 11°15′ | 671 | 56°15′ | 2859 |
| 15°0′ | 890 | 60°0′ | 2978 |
| 18°45′ | 1105 | 63°45′ | 3084 |
| 22°30′ | 1315 | 67°30′ | 3177 |
| 26°15′ | 1520 | 71°15′ | 3256 |
| 30°0′ | 1719 | 75°0′ | 3321 |
| 33°45′ | 1910 | 78°45′ | 3372 |
| 37°30′ | 2093 | 82°30′ | 3409 |
| 41°15′ | 2267 | 86°15′ | 3431 |
| 45°0′ | 2431 | 90°0′ | 3438 |

Aryabhata I fixed the value 3438 for the radius on the basis of his assumption that the circumfrance of a circle in angular measure which should be equivalent to 360° or 360 × 60′ = 21600′ and depending on this value radius should be equal to $\frac{21600}{2(3.1416)} = 3438$ (approx). To find different values in the sine table Aryabhata I used the recursion formula which states that if the $n$th sine in the sequence from $n = 1$ to $n = 24$ are denoted by $s_n$ and if the sum of the first $n$ sines is denoted by $S_n$ then $s_{n+1} = s_n + s_1 - \frac{S_n}{s_1}$. This formula can easily give the value of sin 7° 30′ = 449, sin 11° 15′ = 671, sin 15° = 890 and so on. Aryabhata I also formulated the rules to find the value of sines of allied angles in other quadrants like 90° + θ, 180° + θ and 270° + θ as

$$\sin (90° + \theta) = \sin 90° - \text{vers} (\sin \theta) = \cos \theta$$

$$\sin (180° + \theta) = \sin 90° - \text{vers} \sin 90° - \sin \theta = (- \sin \theta)$$

$$\sin(270^\circ + \theta) = \sin 90^\circ - \text{Vers} \sin 90^\circ - \sin 90^\circ + \text{Vers} \sin \theta = (-\cos \theta)$$

Aryabhata I also introduced many new theorems related to different geometrical figures. Though ancient Hindu mathematicians were aware of the properties of right angled triangled and those were explained in various Sulbasutras but it was Aryabhata I only who gave the rule to find the area of general triangle by expressing that :

त्रिभुजस्य फल शरीरं समदलकोटी भुजार्धसंवर्गः

(A. B. Ganitapada 6)

According to it, the measure of an area of triangle is the product of the perpendicular and half the base. For right angled triangle Aryabhata I expressed that

यश्चैव भुजावर्गः कोटिवर्गश्च कर्ण वर्गः सः

(A. B. Ganitapada 17)

It explains that which is the square of bhuja and that which is the square of koti that is the square of the karna.

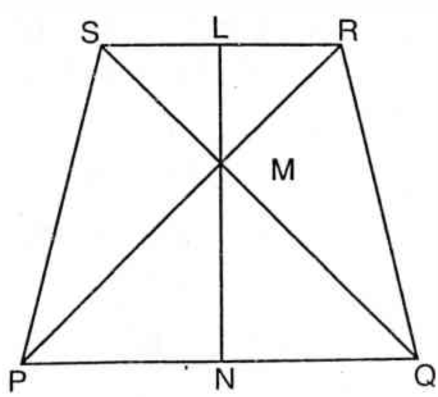
Aryabhata I also gave the rule related to trapezium as

आयामगुणे पार्श्वे तधोगहृते स्वपात रेखे ते

विस्तारयोगार्धगुणे सेयं क्षेत्रफल मायामे

(A. B. Ganitapada 8)

From the rule, we can get




$$LM = \frac{SR}{PQ + SR} \times LN$$

$$MN = \frac{PQ}{PQ + SR} \times LN$$

and area of the figure

$$= LN \times \left(\frac{PQ + SR}{2}\right)$$

To find the area of a circle Aryabhata I expressed that

समपरिणाहस्यार्धं विषकम्भार्धहतमेव वृत्तफलम्

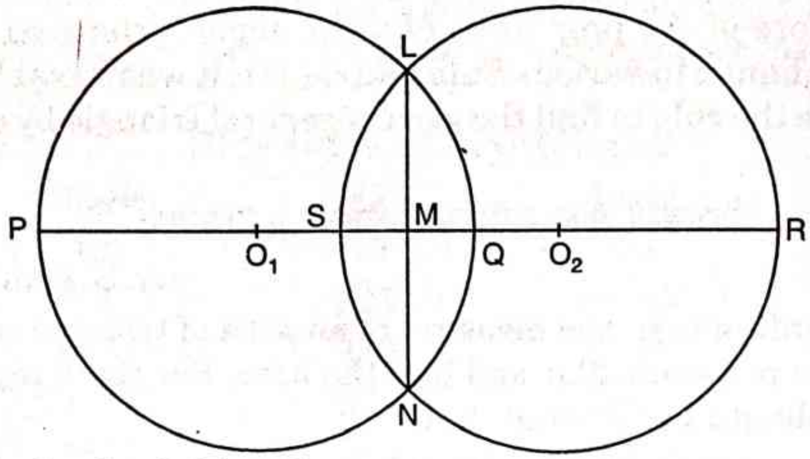
(A. B. Ganitapada)

According to it, area of circle can be found by multiplying half of the circumference with half of the diameter.

He also gave the rule for calculating the heights of the arcs enclosing the common portion of the circles in this sloka :

ग्रासोने द्वे वृत्ते ग्रासगुणे भाजयेत् पृथक्त्वेन
ग्रासोनयोगलब्धौ सम्पात शरौ परस्परतः

(A. B. Ganitapada 18)




On the basis of this rule

$$SM = \frac{(PQ - QS)QS}{(PQ + SR) - 2QS} \quad \text{and} \quad QM = \frac{(SR - QS)QS}{(PQ + SR) - 2QS}$$

Aryabhata I was the first Hindu mathematician who gave the value of $\pi$ correct to four decimal places and very appropriately mentioned that this value of $\pi = 3.1416$ is asanna *i.e.*, approximate value which shows that the value of $\pi$ is incommensurable or irrational. It is highly commendable exposition of Arybahata I, who considered the value of $\pi$ as incommensurable because the same nature of $\pi$ became known to the western mathematicians after thirteen centuries. To compute the value of $\pi$ Arybhata I expressed that

चतुराधिकं शतमष्टगुणं द्वाषष्टि स्तथा सहस्राणाम्
अयुतद्वय विष्कम्भ स्यासन्नो वृत्त परिणाहः

(A. B. Ganitapada 10)

According to this sloka,

(4 + 100) 8 + 62000 = 62832 is the asanna value of the circumference of that circle whose diameter is 20000. As it is known that diameter of any circle is always constant, which is denoted by $\pi$, so

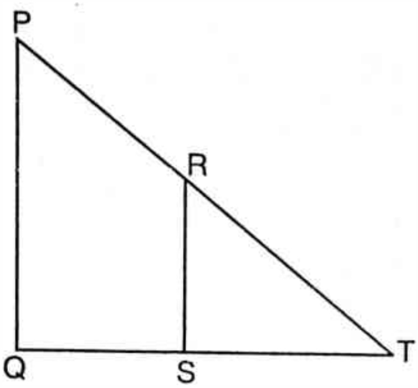
$$\frac{62832}{20000} = 3.1416 = \pi$$

Aryabhata I also dealt with the problems related to gnomon and its shadow. He expressed that

शङ्कुगुणं शङ्कुभुजाविवरं शङ्कुभुजयोर्विशेषहृतम्
यल्लब्धं सा छाया शङ्कोः स्व मूलाद्धि !!

(A. B. Ganitapada 15)

In this sloka Aryabhata I gave the rule to find the following values from the figure.




PQ = the lamp post

RS = the gnomon

ST = the shadow of the gnomon

$$\therefore \quad ST = \frac{QS \times SR}{PQ - RS}$$

In other sloka Aryabhata I explained that

छायागणितं छायाग्रविवरमूनेन भाजिता कोटी
शकंगुणा कोटी सा छायाभक्ता: भुजा भवति

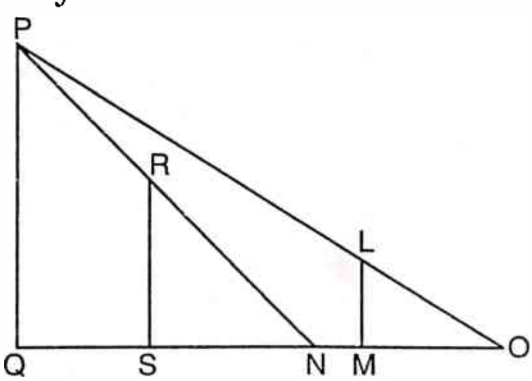
(A. B. Ganitapada 16)

PQ = the lamp post

RS, LM = the gnomon in its two positions

SN, MO = the shadows respectively

then,




$$QN = \frac{NO \times SN}{MO - SN};$$

$$QO = \frac{NO \times MO}{MO - SN}$$

$$PQ = \frac{QN \times RS}{SN}$$

It is noteworthy to mention here that Aryabhata I was the first Hindu astronomer who made it clear that the earth and other planets are not self luminous and they receive and reflect the sun's light. The earth, and the plants are dark due to their own shadows and they become bright because of facing the sun in other halves. He enunciated his own theory about the rising and setting of the sun, the moon and other heavenly bodies by explaining that it is due to the relative motion of the earth, which also causes because of the rotation of the earth about its own axis. He expressed this in the sloka :

अनुलोमगतिर्नौस्थः पश्यत्यचलं विलोमगं यद्वत्
अचलानि भानि तद्वत् समपश्चिमगानि लकांयाम्

(A.B. Golapada 9)

According to it, as a man sitting in the boat moving forward sees the stationary objects moving backward, similarly, stationary stars move exactly towards the west.

Aryabhata I made use of Mahayuga theory which is divided in the ratio 4 : 3 : 2 : 1 called Kreta, Treta, Dvapara and Kali. On the basis of this ratio the duration of Kreta yuga is 1728000 years, Treta yuga is 1296000 years, Dvapara yuga is 8, 64,000 years and Kaliyuga is 432000 years. Which shows that duration of Mahayuga is 4320000 years. For astronomical computations Aryabhata I divided a Mahayuga into four equal parts of equal duartion of 1080000 years. He explained the causes of the eclipses of the sun and the moon correctly and he was the first astronomer who attempted to measure the circumference of the earth and computed it 24835 miles, which is only 67 miles less than the exact value *i.e.* 24,902 miles.

# 6. *Varahamihira*

Varahamihira was a celebrated astrologer, astronomer and mathematician, who was among the nine gems in the court of well known Hindu king Vikramaditya of Gupta dynasty. The other ratnas in his court were Dhanvantari, Ksapanaka, Sanku, Vetalabhatta, Amarsimha, Ghatakharpara, Kalidasa and Vararuci. Varahamihira was the son of Adityadasa and it is believed that he got rudimentary knowledge of astrology and other related areas from his father. Regarding the place of his birth no authentic information is available but we get few ideas about his lineages and place of birth on the basis of few references made in the commentaries of later period written on his works. The great commentator Bhattotpala considered him the incarnation of sun-god and stated that it was the boon of sun-god only which awakened the knowledge of Varahamihira. He also said that Varahamihira was Magadha Dvija. With these informations, some scholars are of the opinion that Varahamihira might have migrated to Ujjaiyini (Avanti) from Magadha.They explained the meaning of Maga as worshipper of the sun. But few other historians opined that Maga-Brahmins migrated to Persia from India and due to those Maga-Brahmins presence in that country the words like Mitra and Aryaman were invoked for sun-god in Persia. In Bhavisya-Purana also we find a sloka related to Magadha, explaining its meaning as "those who meditate on maga (the sun-god)". Besides, these different opinions, few others believe that Varahamihira's ancestors might have migrated from Sakadvipa to their native homeland and due to this lineage Varaha was named as Sakadvipi brahmans whose gotra was Mihira. Again, nothing can be said with full certainity about the year of Varahamihir's birth. Some scholars have fixed his year of birth around 485 A.D., whereas few others fixed it around 505 A.D. According to the statement of Varaha in Brahajjatakam, it is learnt that he was native to Kapithaka and educated by his father Adityadasa during his early age. In the commentary of

Amarajas on Brahmgupta's Khandakhadyaka we find a reference regarding the year of death of Varahamihira as

नवाधिक पञ्चशत् सङ्ख्यशाके
वराहमिहिरचार्यो दिवगंतः

*i.e.* Varahamihira passed away in Saka 509 or 509 + 78 = 587 A.D.

Varahamihira had revolutionized the whole process of thinking about various concepts of mathemetical and astronomical importance. He completely discarded the tradational myths related to the eclipses and explained the scientific cause of an eclipse. He told that when moon enters (covers) the sun's disc, solar eclipse occurs and when moon enters the shadow of the earth, the lunar eclipse occurs. He was the first scholar who expounded that there may be definitely a force which might be keeping heavenly bodies stuck to the round earth. The same force now known as 'gravity'. Varahamihira was a celebrated astronomer and many scholars of later period including Brahmgupta made references about him in their works. Bhaskara II also paid great respect to Varahamihira in his text Siddhanta–Siromani. Varahamihira made important contributions in the areas of mathematics, astronomy and astrology. His important texts are Brihat-Samhita, Brihat Jatak, Laghujatak, Yoga Yatra, Vivaha Patal, Daivaigya-Ballabha, Panca-Siddhantika. It is believed that Varahamihira was able to predict the future events quite early on the basis of his astrological calculations. There is a popular myth about the word 'Varaha' and attachment of this word with his name ; which is also related to his astrological prediction. The king Vikaramaditya once asked Varahamihira to predict about the life of his newly born prince. Surprisingly, enough, Varahamihira made the prediction that prince will have shorter life span and he will die at the hands of a boar (Varaha). Later on, one day the young prince, while playing fell on a stone which was shaped in the form of a boar, the injuries were so fatal that ultimately prince succumbed to them. The prediction of Mihira became true and due to this an unusual epithet was attached with his name "Varaha". Since then, he was called by the name Varahamihira (Kyato-Varahamihiro ; ख्यातो वराहमिहिरो).

Varahamihir's major text related to astoronomy is Panca-Siddhantika. The importance of this text is two fold, firstly, it is a wonderful treatise on mathematical astronomy, secondly, it gives us informations about other older texts written by his predecessor Indian scholars, which are not available now. In this context, it can

be said that Panca Siddhantika is an astronomical text of Varahmihira, which encompasses the related and relevant works of other scholars as well as the works of his own. Presently, this text is taken into consideration as an important source to trace the history of Hindu astronomy upto sixth century CE. In this text Varahamihira summarised five siddhantas namely Paulisha Romaka, Vasistha, Surya and Paitamaha. He included in these Siddhantas the important works of earlier astronomers like Latadeva, Arhat, Aryabhata, Pradyumna, the Magas and the Yamanas. The Romaka–Siddhanta is based on the Greek epicycle theory of the motions of the sun and the moon. In Paitamaha Siddhanta the word "Paitamaha", means grandfather, is devoted to Lord Brahma, who is considered the creater of this universe. In this Siddhanta he took references from Vedanga-Jyotisa of Lagadha. Varahamihira's version of Surya-Siddhanta is condsidered the best among the five siddhantas, which also used astronomical constants similar to those given by Bhaskara I and Brahmgupta in their commentaries and attributed to Arybhata I. In these they considered the start of Yuga at midnight broadly known as Ardharatrika system.

The other important work of Varahamihira is Brhat-Samhita which covers the topics related to Geography, Agriculture, Economics, Physiogamy, Botany, Zoology, Engineering, Erotics, Astronomy etc. It is a compilation of the works done by various other scholars given in different texts and shastras due to which only it is named as Brhat-Samhita *i.e.* great compilation. In Brhat-Simhita he gave elaborate comments about the lunar and scolar eclipses on the basis of scientific principles and denounced them vehemently who explained these events with deep superstitional bias. He made it clear that the eclipses are no way related to any bad omens and other unfavourable indications. In this text he also described four types of earthquakes and their symptoms. These earthquakes he named as Vayumandal ; Agnimandal ; Indramandal and Varunamandal. The third text of Varahamihira is Brhatjatka which is related to astrology. In this text he gave different names for the twelve zodiac signs *i.e.* Kriya, Tavuri, Jituna, Karkin, Heya, Pathena, Juka, Kaurpya, Taukshika, Akokera, Hardorga and Ittha for Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra , Scorpio, Sagittarius, Capricornus, Aquarius and Pisces respectvely. It is also believed that these names given by Varahamihira are of Greek origin, which is sufficient to understand that his text Brhatjatka was upto a great extent influenced by Greek system of knowledge

in the area of astrology. In Pancasiddhanta of Varahamihira we find beautiful place value system of numbers *e.g.*

to express the number 110, he named it by Shunyaikaika

$$= * \text{Shunya} * \text{ eka} *$$
$$= 0 + 1 \times 10 + 1 \times 10^2 = 110$$

to express number 150, he named it by Khatithi = * Kha * tithi

$$= 0 + 15 \times 10 = 150$$

Similarly, he expressed large numbers by this system of numeration. To express a number like 38100, he named it by Khakharupashdaguna = * kha * kha * rupa * ashta * guna

$$= 0 + 0 \times 10 + 1 \times 10^2 + 8 \times 10^3 + 3 \times 10^4 = 38100$$

These examples make it evidently clear that even in the sixth century of Christian era itself Varahamihira was frequently using the place value system with Sanskrit numerals and making use of zero in that system.

Varahamihira made very important contributions in the area of Jyotipatti-ganita presently known as Trigonometry. It is believed that he was the first Hindu mathematician who established following trigonometric identities like

$$\sin^2 x + \cos^2 x = 1$$

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$2\sin^2\left(\frac{x}{2}\right) = 1 - \cos x$$

$$4\sin^2\left(\frac{x}{2}\right) = \sin^2 x + \text{ver}\sin^2 x$$

In Pancasiddanta he gave following formulae :

$$R\sin 30° = \frac{R}{2}\ ;\ R\sin 60° = \frac{\sqrt{3}}{2}R$$

$$R\sin 90° = R = 120'.$$

$$(R\sin A)^2 = \frac{R}{2}(R - R\cos 2A)^2$$

$$(R\sin A)^2 + (R\cos A)^2 = R^2$$

$$\text{jya}\left(\frac{\pi}{2} - A\right) = \sqrt{R^2 - \text{jya}^2 A}$$

$$(\text{jya A})^2 = \frac{R}{2}\,[R - \text{kojya } 2A]$$

$$(\text{jya A})^2 = \left[\frac{\text{jya}(2A)}{2}\right]^2 + \left[\frac{R - \text{jya}(\pi/2 - 2A)}{2}\right]^2$$

Varahamihira also devised sine tables by improving the tables of Aryabhata I and found more accurate values from his tables. His table contains the value of R sines from 3° 45′ to its 24 multiplies. He started his calculation first with the known values of R sin 30°, R sin 45°, and R sin 60° by using repeated application of the formulae

$$\sin\left(\frac{x}{2}\right) = \frac{1}{2}\sqrt{\sin^2 x + \text{ver}\sin^2 x}$$

and $$\sin\left(\frac{x}{2}\right) = \sqrt{\frac{1}{2}\text{ver}\sin x}$$

Varahamihira also constructed the array to find the combinatorial value of $^nC_r$. He arranged the numbers $n$ in a column with $n = 1$ at the bottom and then placed the numbers $r$ in the rows with $r = 1$ at the left hand side. After arranging these numbers in this way the value of $^nC_r$ is obtained by summing two entries *i.e.* the one directly below the $(n, r)$ position and the other entry immediate left to it. This array seems to be more or less similar to the Pascal's triangle for finding the bionomial coefficients. Varahamihra also constructed pan diagonal magic squares.

Varahamihira is known now a mathematician and astronomer of scientific outlook, who attacked all those myths, popular beliefs and superstitions, which prevailed in the name of astrology or astronomy. He devoted his life in finding the true causes of various incidents related to heavenly bodies, their movements, mateorological phenomena and number of other concepts of astronomical importance. About him Al-Biruni rightly pointed out that "Varahamihira has revealed himself to us as a man who accurately knows the shape of the world. On the whole, his foot stands firmly on the basis of truth and he cleverly speaks out the truth.

# 7. Bhaskara I

Bhaskara I was ancient Hindu mathematician who probably lived during sixth and seventh centuries after the Christian era and wrote commentaries on the works of Aryabhata I. He left little evidences in his texts to ascertain the facts related to his birth place and life span. But some inferences are drawn by the scholars on the basis of available literature of that time and references made by Astronomers of later period about Bhaskara I. K.S. Shukla presented his views in this regard that Bhaskara I was probably working in a school of mathematics at Asmaka, which was situated in Andhra Pradesh. It seems that the mathematicians of this school were the followers of Aryabhata I. We also find some references in the texts of Bhaskara I about the places Valabhi ; the capital city of the Maitraka Dynasty of the seventh century A.D. and Sivrajpura ; These both the places were situated in the Saurastra area of modern Gujrat state. There are some other references of the places like Bharuch (a place in Southern Gujrat) and Thaneswar (a place in Eastern Punjab). Considering the references of these places also, some scholars are of the opininon that probably Bhaskara I was born in Saurastra area and later on moved to Asmaka in Andhra Pradesh. Bhaskara I gave the reference of Asmaka at several places in his texts and named the scholars of Asmaka as Asmakiyas. Bhaskara I wrote three mathematical texts *viz* Maha-Bhaskariyam, Aryabhatiya Vyakhya and Laghu-Bhaskariyam, which are commentaries on the works of Aryabhata I. He elaborately explained the astronomical concepts of Aryabhata I and covered the development of these concepts till the time of his writings in these commentaries. These texts of Bhaskara I were followed by scholars in different parts of India for a long period and many commentaries on these texts were also given by the scholars like Govindasvamin, Sankaranarayana, Parameswara etc. The commentary of Bhaskara I on Aryabhatiya is divided into two sections namely Rasiganita

(mathematics of numbers) and Ksetraganita (mathematics of figures). He included the topics like proportion and kuttaka (indeterminate analyasis) in Rasiganita whereas the topics like series, shadow problems etc. in Ksetraganita. Bhaskara I also quoted three Prakrtgathas in his commentary and mentioned the names of four Jaina mathematicians Maskari, Purana, Mudgals and Putana also in it.

Bhaskara I gave full description of decimal arithmetic in his commentary on Aryabhatiya. It is believed that the earliest record of the use of place value system belongs to sixth centry A.D. and Bhaskara I also used this concept to represent any number by using letter numerals. He also gave the name kha for zero and used zero to denote the notational places. He used the numbers in word symbols and figures. He followed the nine basic numerals of decimal place value system. We express below few numbers in word symbols and their conversion into nine basic numerals :

First number : 432 00 00

Word symbol for this number is :

Viyad . ambara, akasha, sunya, yama, rama, veda.

where, viyad → sky → 0
ambara → atmosphere → 0
akasha → ether → 0
skunya → void → 0
yama → primordial → 2
rama → Rama → 3
veda → veda → 4

It can be expressed in figures as

00 00 234

or $0 + 0 \times 10 + 0 \times 10^2 + 0 \times 10^3 + 2 \times 10^4 + 3 \times 10^5 + 4 \times 10^6$
$= 432\ 00\ 00$

Second number : 1986123730

Word symbol for this number is :

kha . agny . adri . rama . arka . rasa . vasu . randhra . indavah

where, kha → space → 0
agny → fire → 3
adri → mountains → 7
rama → Rama → 3
arka → Sun → 12

rasa → Senses → 6
vasu → Vasu → 8
randhra → Orifics → 9
indavah → moon → 1

It can be expressed in figures as

0 3 7 3 12 6 8 9 1

or, $0 + 3 \times 10 + 7 \times 10^2 + 3 \times 10^3 + 12 \times 10^4 + 6 \times 10^6 + 8 \times 10^7 + 9 \times 10^8 + 1 \times 10^9$

$= 1986123730$

Bhaskara I in his commentaries covered the topics related to astronomy like longitudes of the planets, lunar crescent eclipses of the sun and the moon etc. He found the approximate value of sine function by using rational fractions as

$$\sin A = \frac{16A(\pi - A)}{5\pi^2 - 4A(\pi - A)}$$

where angle A is measured in radian and from this formula the values for different angles can also be obtained like

$$\sin\left(\frac{\pi}{4}\right) = 0.70558 ;$$

$$\sin\left(\frac{\pi}{7}\right) = 0.4313$$

$$\sin\left(\frac{2\pi}{5}\right) = 0.95050$$

..................................

..................................

These values are correct upto two places of decimals. But when this angle A is measured in degrees the same formula takes another form as,

$$R \sin A = \frac{4R\,(180° - A)A}{[40500 - (180° - A)A]}$$

Bhaskara I also established the trignometric identities to find the value of sine function, when the measure to given angle is greater than 90° as

$$R \sin (90° + \theta) = R \sin 90° - R \text{ ver} \sin \theta$$

$$R \sin (180° + \theta) = R \sin 90° - R \text{ ver} \sin (90°) - R \sin 90°$$

$$= - R \sin \theta$$

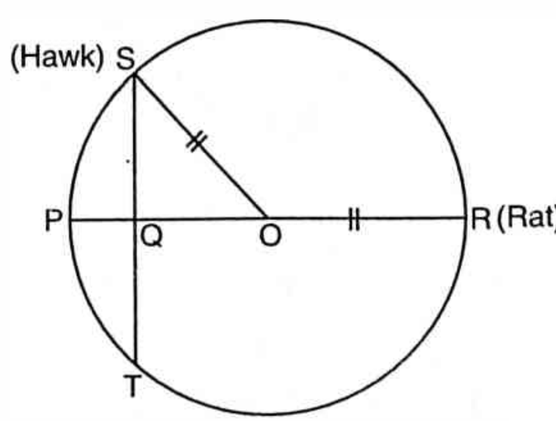
and so on.

Bhaskara I also covered the topics related to algebra like solution of linear equations, quadratic equations, cubic equations and equations with more than one variable. It is considered that his commentary on Aryabhatiya is the first treatise of mathematics where we find the geometrical treatment of different algebraic expressions, formulae and series. He explained the method to find the solution in positive integers for the indeterminate equations of the first degree like $by - ax = -c$ and $by - ax = -1$. Bhaskara I also gave the solutions of the problems in which three or more functions either linear or quadratic of the given variables to be made squares or cubes. In his text Laghu-Bhaskariya we find a problem of this type *i.e.* "find two numbers $x$ and $y$ such that the expressions $x + y$, $x - y$, $xy + 1$ all are perfect squares". Bhaskara I also solved problems related to geometry. He beautifully formulated the area of triangle where altitude is not known and found the area in the form of square root of the product of $s$, $(s - a)$ $(s - b)$ and $(s - c)$, taking $s$ as semiperimeter and $a$, $b$ and $c$ the sides of given triangle. To obtain this formula he expressed altitude of triangle in terms of the sides and segments of base calling them as abadha and abadhantara) and thereafter substituting these values in the formula given by Aryabhata I *i.e.* area of triangle $= \frac{1}{2} \times$ base $\times$ altitude.

Now we give two famous problems of Bhaskara I, generally known as Hawk-Rat problem and Crane-Fish problem.

(1) **Hawk-Rat problem.** A hawk is sitting on a pole whose height is 18 cubits. A rate which had gone out of its dwelling, at the foot of the pole, to a distance of 81 cubits, while returning to its dwelling is killed by the cruel hawk on the way. Say, how far has it gone towards the hole and also the horizontal motion (the speeds of the rat and hawk being the same)

(Tr. by Dr. S.B. Rao)




Let hawk be at S and dwelling be at Q. Rat follows the path QOR and hawk the path SO. Since the speeds of rate and hawk are same, so SO = OR

Now, from the figure

$$SQ.\ QT = PQ\ .\ QR$$

$$\Rightarrow \qquad SQ^2 = 81PQ$$

or $$PQ = \frac{18 \times 18}{81} = 4 \text{ cubits}$$

Again, from the figure

$$QO = QR - RO$$
$$= QR - PO$$
$$= QR - (PQ + QO)$$

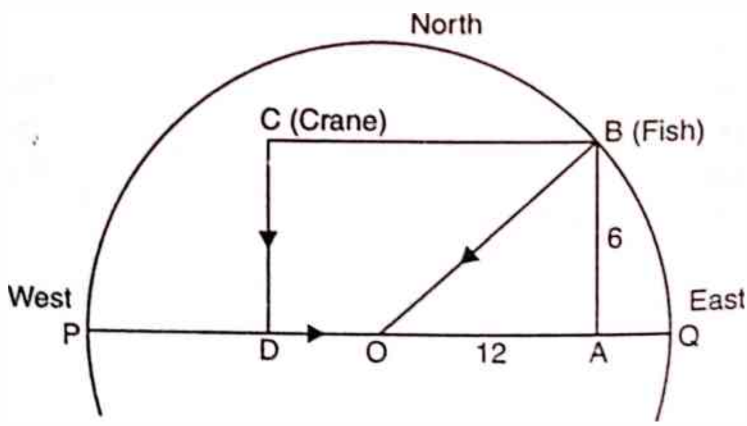
or $$QO = \frac{1}{2}(QR - PQ) = \frac{1}{2}(81 - 4)$$
$$= 38\frac{1}{2} \text{ cubit}$$

Similarly, we get

$$OR = \frac{1}{2}(QR + PQ) = \frac{1}{2}(85) = 42\frac{1}{2} \text{ cubits.}$$

(*ii*) **Crane-Fish Problem.** There is a reservoir of water of dimensions 6 × 12. At the north-east corner of the reservoir there is a fish, and at the north-west corner there is a curve. For the fear of the crane the fish, crossing the reservoir hurriedly went towards the south in an oblique direction but was killed by the crane who came along the sides of the reservoir. Give out the distances travelled by them. (assuming that their speeds are same)

(Tr. by Dr. S.B. Rao)




In above figure ABCD is reservoir of rectangular shape. The fish is at B (north-east corner) and crane is at C (north-west corner). Fish swims through BO towards south and reaches at O, at the same time interval crane briskly moves through CD and DO to reach at the point O to attack the fish. Since the speeds of the crane and fish are same, no fish and crane travel equal distances in the

same interval of time, which give

$$BO = CD + DO$$

Now extend OD upto point P such that OP = OB, and from the figure

$$AP = AD + DP$$

but $\quad CD + DO = OB \quad$ and $\quad OB = OP$

so, $\quad CD + DO = OP \quad$ and $\quad CD = OP - DO = PD$

and $\quad AP = AD + CD = 12 + 6 = 18$

Again extend PA to Q, from which we get

$$QA \,.\, AP = AB^2$$

and $\quad QA = \dfrac{AB^2}{AP} = \dfrac{36}{18} = 2$

Further, $\quad QP = QA + AP = 2 + 18 = 20$

and $\quad OP = OB = 10.$

It shows that crane and fish both travel distance of 10 units each.

Bhaskara I was also familiar with the proposition that if $p$ is a prime number then $1 + (p - 1)\,!$ is divisible by $p$ and also gave the solution to the problems like "what should be value of $x$, whose square multiplied by 8 and then added to 1, again is a square". In modern notation this equation can be expressed as $8x^2 + 1 = y^2$ and its solutions are $x = 1, y = 3$ ; $x = 6, y = 17$ etc.

# 8. *Brahmagupta*

Year of Birth : 598 C.E.
Year of Death : 670 C.E. (Probably)

Brahmagupta was the most prominent mathematician of seventh century A.D., who belonged to Ujjain school and lived during the period of Srivyaghramukha, the greatest king of Capa dynasty. His father was Jishnugupta and he was resident of Bhinamala, a place situated in the South Marwar area of Gujarat. Bhinamala was also the capital of North Gujarat during seventh century and due to his nativity with it, one of his commentators, Varuna referred him as Bhinamalakacarya. Brahmgupta stated that he authored his celebrated text Brahma–Sphuta–Siddhanta in Saka year 550 *i.e.* 628 C.E., when he was thirty years old, which made it clear that this great mathematician and astronomer born in 598 C.E. It is described by him in following sloka :

श्री चापवशं तिलके श्री व्याघ्रमुखे नृपे शकनृपाणाम
पंचाशत्सयुक्तैवर्ष शतैः पर्चभिरतोते ।। 7 ।।
ब्राह्मः स्फुटसिद्धान्तः सज्जनगणितज्ञ गोल वित्प्रीत्यै
त्रिशंद्वर्षेण कृतो जिष्णुसुत ब्रह्मगुप्तेन ।। 8 ।।

"संज्ञाध्याय, ब्रह्मस्फुटसिद्धान्त"

The important texts written by Brahmagupta are Brahma-Sphuta-Siddhanta and Khandakhadyaka. Though it is also believed by some scholars that he authored few more slokas under title Dhyanagraha, which were not included in these two texts but these slokas could not be traced even till this time. Brahma-Sphuta-Siddhanta contains 1008 verses spread into 24 chapters. The first ten chapters of this text cover same topics which were also given in other Siddhantas. The rest of 14 chapters are devoted to those important topics which were not covered in other Siddhantas. The twelfth chapter of this text covers topics related to arithmetic

twelfth chapter of this text covers topics related to arithmeti mensuration etc, which are expressed in fity six verses. Th eighteenth chapter deals with algebra, which Brahmagupta denote by the name Kuttaka and contains 102 verses. It is interesting t note that Brahmagupta was the first Hindu mathematician, wh treated arithmetic and algebra as two seperate branches c mathematics. He expressed his own rules on these topics an compared them with the rules given by his predecessors. Due t such speciality, his texts are of high historical importance to ge authentic informations about the conceptual development of variou topics upto his time and contributions made by othe mathemeticians and astronomers till that time. In Ganitadhyaya we find rules related to cyclic triangles and quadrilaterals, rule for arithmatical operations involving zero and negative number etc. The Kuttakadhyaya presents the solutions of the indeterminat equations of first and second degree. His second tex "Khandakhadyaka," which literally means a particular type of swee prepared by sugarcane, covers the topics like the longitudes of th planets, the three problems of diurnal rotation, solar and luna eclipses, conjunctions of the planets and the moons crescent etc. I chapter IX of this text he gave a method to find the sines c intermediate angles from given sine table. He used the interpolatio formula to compute values of sines, which is a particular case upt second order of Newton–Stirling interpolation formula.

Khandakhadyaka is divided into two parts Proper and Uttara where proper part consists of 194 verses and uttara part consists c 71 verses. It seems that Brahmagupta authored Khandakhadyak in 665 A.D. In the beginning of this text he described that "I compil the work Khandakhadyaka, which gives results to those given b the great scholar Aryabhata. Since it is impossible to carry on ever day affairs with Aryabhata's work, this work is being compiled s as to give easily, equally accurate results relating to matters lik birth, marriage and the like". (Tr. by S. Balachandra Rao

Brahmagupta adopted place value system using the nin numerals and zeros along with the Sanskrit numerical symbols He described twenty logistics and eight determinants in Patiganita Prthudakasvami, one of the commentators of Brahma–Sphuta Siddhanta classified these logistics and determinants as logistics [1. Samkalita (Addition) 2. Vyavakalita (Subtraction) 3. Gunana (Multiplication) 4. Bhagahara (Division) 5. Varga (Square) 6. Vargamula (Square root) 7. Ghana (Cube) 8. Ghanamula (Cube root) 9–13. Panca–jati (the five rules related to five standard forms

of fractions) 14. Trairasika (The Rule of Three), 15. Vyasta Trairasika (Inverse Rule of Three), 16. Pancarasika (The Rule of Five) 17. Saptarasika (The Rule of Seven) 18. Navarasika (The Rule of Nine) 19. Ekadasarasika (The Rule of Eleven). 20. Bhanda-Pratibhanda (Barter and Exchange)] ; Determinants, [1. Misraka (Mixture) 2. Sredhi (series) 3. Ksetra (plane figures) 4. Khata (Excavation), 5. Citi (stock) 6. Krakacika (Saw) 7.Rasi (Mound) 8. Chaya (shadow)].

Brahmagupta described four methods of multiplication namely, Gomutrika, Khanda, Bheda and Ista. He also explained the methods to find division, square, square root, cube, cuberoot etc. He followed the identity $x^2 = (x + a)(x - a) + a^2$ ; to get the value of square of any given number. He considered only five kinds of fractions *i.e.* jatis namely Bhaga, Prabhaga, Bhaganubandha, Bhagapavaha and Bhaga-Bhaga and presented the rules for applying various operations on them under the title Bhagajati. He also explained the rule of three, inverse rule of three etc. Brahmagupta was the first Hindu mathematician, who devised the rules for operation of zero with other numbers. He gave arithmatic rules considering fortunes (positive number) and debts (negative number) as follows :

(*i*) a debt minus zero is a debt

*i.e.* $(-a) - 0 = (-a)$

(*ii*) a fortune minus zero is a fortune

*i.e.* $a - 0 = a$

(*iii*) zero minus zero is a zero.

*i.e.* $0 - 0 = 0$

(*iv*) a debt subtracted from zero is a fortune

*i.e.* $0 - (-a) = a$

(*v*) a fortune subtracted from zero is a debt

*i.e.* $0 - a = (-a)$

(*vi*) the product of zero multiplied by a debt or fortune is zero

*i.e.* $a \times 0 = 0$ ; $(-a) \times 0 = 0$

(*vii*) the product of zero multiplied by zero is zero

*i.e.*, $0 \times 0 = 0$

He further stated that zero divided by zero is nought *i.e.* $0 \div 0 = 0$ ; positive or negative divided by cipher is taccheda. The first part of this statement is not true but second part *i.e.* $a \div 0$ = taccheda, is correct because here the meaning of taccheda is kha-ccheda, which is infinite.

Kuttaka-Ganita, where Kuttaka means pulveriser. He expressed higher powers than the fourth of any unknown quantity or avyakta by using the term 'gata' *e.g.* the fifth power as pancagata, the sixth power as sadgata, similarly, the term for any power by adding the name of number indicating the power with gata. Interestingly enough, he used Varna to symbolise unknowns as Kalaka(black), nilaka (blue), pitaka (yellow) and haritaka (green) and to simplify the representation of these varnas, he only used the first letter of these varnas like k, n, p, h .... It seems that algebra of syncopated where addition was shown by juxtaposition, subtraction by putting dot over the subtrahend, division by placing the divisor below the dividend, the operations of multiplication and evolution as well as unknown quantities were denoted in abbreviated form by using proper words. He explicitly described the laws of signs, as mentioned below :

(*i*) the sum of two positive numbers is positive ; of two negative numbers is negative ; of a positive and a negative number is their difference

(*ii*) from the greater should be subtracted the smaller ; the final result is positive ; if positive from positive and negative, if negative from negative

(*iii*) the product of a positive and a negative is negative, of two negatives, is positive, positive multiplied by positive is positive

(*iv*) positive divided by positive or negative divided by negative becomes positive ; but positive divided by negative is negative and negative divided by positive remains negative

(*v*) the square of a positive or a negative number is positive .... The (sign of the) root is same as was that from which the square was derived

(*vi*) the product of two like unknowns is a square ; the product of three or more like unknowns is a power of that designation (*i* to *vi* translated by Datta & Singh)

Brahmagupta classified equations as equations in one unknown (eka – varna samikarana) ; equations in several unknowns (aneka – varna samikarana) ; equations involving products of unknowns (bhavita). These equations were further classified in linear and quadratic equations involving one, two or several unknowns. To solve linear equations in one unknown he clarified the rule that the value of the unknown should be the quotient of the difference

the value of the unknown should be the quotient of the difference of known terms taken in reverse order with the difference of the coefficients of the unknown. For linear equations with two unknowns he stated that the sum is to be increased and decreased by the difference and divided by two. He also gave rule for solving linear equations in several unknowns *e.g.* for the equation

$$\Sigma x \pm cx_1 = a_1 \,;\, \Sigma x \pm cx_2 = a_2 \,;\; \ldots.. \,\Sigma x \pm cx_n = a_n$$

$$\Sigma x = \frac{a_1 + a_2 + \ldots + a_n}{n \pm c}$$

and

$$x_1 = \frac{1}{c}\left[\pm a_1 \mp \left(\frac{a_1 + a_2 + \ldots + a_n}{n \pm c}\right)\right]$$

$$x_2 = \frac{1}{c}\left[\pm a_2 \pm \left(\frac{a_1 + a_2 + \ldots + a_n}{n \pm c}\right)\right]$$

and so on.

He gave two specific rules for solving the quadratic equation of the form $ax^2 + bx = c$ ; as

$$(i)\; x = \frac{\sqrt{4ac + b^2} - b}{2a}$$

$$(ii)\; x = \frac{\sqrt{ac + \left(\frac{b}{2}\right)^2} - \left(\frac{b}{2}\right)}{a}.$$

It seems that the Brahmagupta was aware about the existence of two roots of quadratic equation. He also gave solutions for simultaneous quadratic equations.

Brahmagupta dealt with indeterminate equations of first and second degree of the forms $by - ax = \pm c$ or $by + ax = \pm c$ and $Nx^2 + 1 = y^2$ or $Nx^2 \pm c = y^2$. He also gave solutions to solve simultaneous indeterminate equations of the first degree *i.e.* $N = a_1 x_1 + r_1 = a_2 x_2 + r_2 = a_3 x_3 + r_3 \ldots.. = a_n x_n + r_n$. He was the first mathematician, who gave solutions in rational integers of Varga-Prakrti type of equations *i.e.* indeterminate equations of second degree as $Nx^2 \pm c = y^2$ or $Nx^2 \pm 1 = y^2$. The equation $Nx^2 + 1 = y^2$ is wrongly named as Pells equation (1668 C.E.) by Western Historians because it was solved even nine centuries before by Brahmagupta and Brahmagupta rightly deserves to get the credit for the same. This type of equations were called by Hindu mathematicians as Varga-Prakriti, where Varga means square and refer to $x^2$ in the equation $Nx^2 + 1 = y^2$.

and Prakriti refers to coefficient N. Brahmagupta gave the word 'gunaka' for the coefficient N. He established two lemmas in order to get the solution of the equation of form $Nx^2 + 1 = y^2$ which are given below.

**Brahmagupta's Lemmas :** Suppose $x = l_1$ and $y = m_1$ are solutions of the equation $Nx^2 + p = y^2$ and $x = l_2$ and $y = m_2$ are solutions of the equation $Nx^2 + p_1 = y^2$, then according to Brahmagupta $x = l_1m_2 \pm l_2 m_1$ and $y = m_1 m_2 \pm N l_1l_2$ is solution of the equation $Nx^2 \pm pp_1 = y^2$ , which shows that if $Nl_1^2 \pm p = m_1^2$ and $Nl_2^2 \pm p_1 = m_2^2$ then,

$N(l_1m_2 \pm l_2m_1)^2 + pp_1 = (m_1m_2 \pm Nl_1l_2)^2$ particularly if $l_1 = l_2$ ; $m_1 = m_2$ and $p = p_1$ then Brahmagupta finds another solution $x = 2l_1 m_1$ ; $y = m_1^2 + Nl_1^2$ of the equation $Nx^2 + p = y^2$, from known solution $x = l_1$ and $y = m_1$. It can also be explained in other words as,

if $Nl_1^2 + p = m_1^2$, then

$$N(2l_1m_1)^2 + p = (m_1^2 + Nl_1^2)^2$$

It is highly interesting that these lemmas of Brahmagupta were rediscovered by Euler in 1764 C.E. and by Lagrange in 1768 C.E. almost eleven centuries after the period of Brahmagupta. Brahmagupta named these lemmas as Bhavana and classified them according to the operations addition and subtraction with the names Samasa Bhavana and Antara Bhavana respectively. Further, if composition is done with equal sets of roots then called as Tulya Bhavana and if composition is with unequal sets of roots then called as Atulya Bhavana. According to the Brahmaguptas lemmas, if $(l_1, m_1)$ and $(l_2, m_2)$ are two solutions of the equation $Nx^2 + 1 = y^2$, then $x = l_1m_2 \pm l_2m_1$ ; $y = m_1m_2 \pm Nl_1 l_2$ are two other solutions. Further by applying particular result *i.e.* if $(l_1, m_1)$ is a solution of $Nx^2 + 1 = y^2$, then another solution is $(2l_1 m_1, m_1^2 + Nl_1^2)$ and continuing this process repeatedly, the infinitely many solutions of this equation can be obtained. Brahmagupta further stated that if $(l, m)$ be the solution of equation $Nx^2 + p^2 = y^2$, then $x = \frac{l}{p}$ and $y = \frac{m}{p}$ should be the solution for the equation $Nx^2 + 1 = y^2$. This result was also explained in slightly different manner for the equation of the form $Nx^2 \pm p^2k = y^2$ ; because if we put here, $u = \frac{x}{p}$ and $v = \frac{y}{p}$, then the equation $Nx^2 \pm p^2k = y^2$, get reduced to another form $Nu^2 \pm k = v^2$, which shows that the roots of the original equation are $p$ times of the newly formed equation. By applying this result

the solutions of the equations like $6x^2 + 12 = y^2$ ; $6x^2 + 75 = y^2$ and $6x^2 + 300 = y^2$ can be obtained from the equation $6x^2 + 3 = y^2$ ; because $12 = 4.3 = 2^2.3$ ; $75 = 25.3 = 5^2 . 3$ ; $300 = 100 . 3 = 10^2 . 3$.

Brahmagupta also made it clear that for the equation $Nx^2 + p = y^2$, an integral solution can always be obtained if $p = \pm 1$ or $+ 2$ or $\pm 4$ which may further generate infinitely many solutions by repeated application of Samasa Bhavana. He also gave rational solution for Varga-Prakarti $Nx^2 + 1 = y^2$. By choosing conveniently P and $P_1$, if $(l, m)$ and $(l_1, m_1)$ are sets of solutions of $Nx^2 + P = y^2$ and $Nx^2 + P_1 = y^2$, then $N(lm_1 \pm l_1 m)^2 + PP_1 = (mm_1 \pm Nll_1)^2$

Thus,

$$N\left(\frac{lm_1 \pm l_1 m}{\sqrt{PP_1}}\right)^2 + 1 = \left(\frac{mm_1 \pm Nll_1}{\sqrt{PP_1}}\right)^2$$

here, if $PP_1$ is perfect square, then

$$x = \frac{lm_1 \pm l_1 m}{\sqrt{PP_1}} \; ; y = \frac{mm_1 \pm Nll_1}{\sqrt{PP_1}}$$

are rational solutions of $Nx^2 + 1 = y^2$. If $PP_1$ is not prefect square then $Nx^2 + P_1 = y^2$ is to be replaced by $Nx^2 + P = y^2$. In particular, if

$P = P_1$ then roots become $\dfrac{lm \pm lm}{P}$ and $\pm\left(\dfrac{m^2 \pm Nl^2}{P}\right)$.

Brahmagupta also presented solutions for following type of equations :

(*i*) $Nx^2 + c = y^2$

If $(k, l)$ be any rational solution and $(m, n)$ be a solution of equation $Nx^2 + 1 = y^2$ then by applying the Samasa, $x = kn \pm lm$, $y = ln \pm Nkm$ is a solution of given equation and by repeated application of this process, infinitely many solutions can be obtained.

(*ii*) $Mn^2x^2 + c = y^2$

Put $nx = z$, then given equation is transformed to $Mz^2 + c = y^2$

and if $(z, y)$ is rational solution of this equation, then $\left(\dfrac{z}{n}, y\right)$ should be rational solution of the equation $Mn^2x^2 + c = y^2$

(*iii*) $a^2x^2 \pm c = y^2$

Put it in the form $y^2 - a^2x^2 = \pm c$

or $$(y + ax)(y - ax) = \pm c$$

Let $\qquad y - ax = p$, so $\quad y + ax = \pm \dfrac{c}{p}$

Thus, $$x = \frac{1}{2a}\left(\pm \frac{c}{p} - p\right)$$

and $\quad y = \dfrac{1}{2}\left(\pm \dfrac{c}{p} - p\right)$ are rational solutions of given equation, where $p$ is any rational number.

We give here some of the examples related to indeterminate quadratic equations which were solved by Brahmagupta.

(*i*) For the equation $8x^2 + 1 = y^2$, he gave the solutions $(x, y) = (1, 3)$ ; $(6, 17)$ ; $(35, 99)$ ; $(204, 577)$ , $(1189, 3363)$ ......

(*ii*) For the equation $11x^2 + 1 = y^2$, he gave the solutions $(x, y) = (3, 10), \left(\dfrac{161}{5}, \dfrac{534}{5}\right)$ .....

How interesting it is, that the smallest values for the solutions of $61x^2 + 1 = y^2$, he found as $x = 226153980$ ; $y = 1766319049$.

Brahmagupta also gave solutions of the square pulverisers of the form $bx + c = y^2$ and solved the simultaneous indeterminate quadratic equation of the type $x \pm p = y^2$ and $x \pm q = y^2$, for which he gave the solution as

$$x = \left\{\frac{1}{2}\left(\frac{p-q}{m} \pm m\right)\right\}^2 \mp p$$

or $$x = \left\{\frac{1}{2}\left(\frac{p-q}{m} \mp m\right)\right\}^2 \mp p.$$

He also gave rule for the particular case of this type of equations as $x + p = y^2$ and $x - q = z^2$ $\quad$ as $x = \left\{\dfrac{1}{2}\left(\dfrac{p+q}{m} - m\right)\right\}^2 + p$

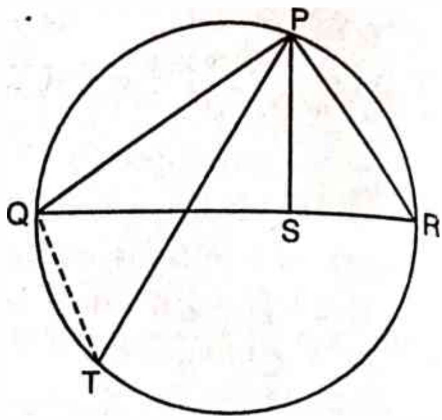
Brahmagupta's contribution in the field of geometry is remarkable. He presented many new theorems related to various plane figures. To find the circumradius of a triangle he explained the rule in this sloka as

त्रिभुजस्य वधो भुजयोर्द्विगुणितलम्बोद्धृतो हृदय-रज्जुः

सा द्विगुणा त्रिचतुर्भुजकोण स्पृग्वृत्त विष्कम्भः

(Br. Sp. Sid. XII 27)

If $r$ is circumradius of the triangle PQR, then $r = \frac{PQ \cdot PR}{2PS}$ where, PS is altitude of triangle.




This result can also be proved by drawing PT diameter and completing the triangle PQT. Then triangles PQT and PSR will be similar, which shows that

$$\frac{PQ}{PS} = \frac{PT}{PR} \quad \text{or} \quad PT = \frac{PQ \cdot PR}{PS}$$

but PT is diameter, so

$$PT = 2r = \frac{PQ \cdot PR}{PS} \quad \text{or} \quad r = \frac{PQ \cdot PR}{2PS}$$

Brahmagupta also gave rules to construct different type of triangles *e.g.* for right triangle he stated that if $l$, $m$ are any two rational numbers then the sides of right triangle should be $\left\{l, \frac{1}{2}\left(\frac{l^2}{m} - m\right), \frac{1}{2}\left(\frac{l^2}{m} + m\right)\right\}$. Brahmagupta was the first mathematician who gave solution for the equation $x^2 + y^2 = z^2$ in integers as $l^2 - m^2$, $2lm$, $l^2 + m^2$, for any two unequal integers. Further, to find all rational isosceles triangles having the same altitude, he gave the values of the sides and bases of all such rational isosceles triangles as $\frac{1}{2}\left(\frac{h^2}{l} + l\right)$ ; $\frac{1}{2}\left(\frac{h^2}{l} + l\right)$ and $\left(\frac{h^2 - l}{l}\right)$, where $h$ is altitude and $l$ any rational number. Similarly, to find the sides of any rational scalene triangle he gave the value of sides as $\frac{1}{2}\left(\frac{l^2}{m} + m\right)$ ; $\frac{1}{2}\left(\frac{l^2}{n} + n\right)$ ; $\left\{\frac{1}{2}\left(\frac{l^2}{m} - m\right) + \frac{1}{2}\left(\frac{l^2}{n} - n\right)\right\}$ for any rational numbers $l$, $m$, and $n$. It is also interesting to note that Brahmagupta presented the solutions of the rational scalene triangle by juxtaposing two rational right triangles with common leg. This method was discovered in Europe by Bachet (1621 C.E.) almost after one thousand years.

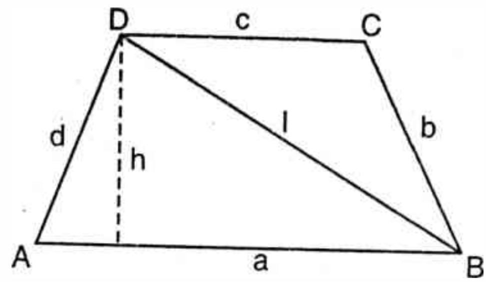
Brahmagupta classified quadrilaterals into five forms namely sama—caturasara (square), ayata—caturasara (rectangle), dvisama—caturasara (isosceles trapezium), trisama—caturasara

(trapezium with three equal sides) and visama—caturasara (quadrilatedral of unequal sides). Brahmgupta enunciated several rules related to quadrilateral, some of them are given below :

(*i*) अविषमचतुरस्र भुजप्रति भुजवधयोर्युतेः पदं कर्णः

(Br. Sp. Si XII 23)

According to the commentator of Brahmagupta's texts, the meaning of avisama in this vevse includes the square, the rectangle and the trapezium. So, if, $a, b, c$ and $d$ are the sides of any isosceles trapezium and $l$ the length of diagonal then




$$l = \sqrt{ac + bd}$$

if $b = d$, then $$l = \sqrt{ac + b^2}$$

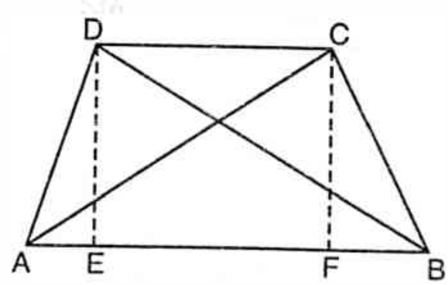
and $$h = \sqrt{l^2 - \left(\frac{a+c}{2}\right)^2}$$

(*ii*) To find the circumradius of an isosceles trapezium he expressed that

अविषमपार्श्व भुजगुणः कर्णो द्विगुणावलम्बकविभक्तः हृदयं ......

(Br. Sp. Si XII 26)

According to this sloka,




$$\text{circumradius} = \frac{BD \times AD}{2DE}$$

(*iii*) To find the area of quadrilateral, Brahmagupta, beautifully expressed the rule in this sloka

स्थूलफलं त्रिचतुर्भुज बाहुप्रतिबाहु योग दल घातः
भुजायोगार्धचतुष्टय भुजोनघातात् पदं सूक्ष्मम्॥

(Br. Sp. Si. XII 21)

According to this sloka, if $a$, $b$, $c$ and $d$ are sides of a quadrilateral then its area should be

$$\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where $s$ is semiperimeter and equal to $\frac{1}{2}(a + b + c + d)$. This is a

remarkable result of Brahmagupta, which generalises the result of Heron, but this formula is applicable only for cyclic quadrilaterals.

(*iv*) Brahmagupta described the rule to find the diagonals of a cyclic quadrilateral in this sloka

कर्णाश्रित भुजघातैक्यमुभयथान्योन्यभाजितं गुणयेत्
योगेन भुजप्रतिभुजवधयोः कर्णौ पदे विषमे॥

(Br. Sp. Si XII 28)

From this rule, for any cyclic quadrilateral of sides $a, b, c$ and $d$ and diagonals $l$ and $m$, the values of $l$ and $m$ are

$$l = \sqrt{\frac{(ab+cd)(ac+bd)}{ad+bc}}, \quad m = \sqrt{\frac{(ad+bc)(ac+bd)}{(ab+cd)}}$$

This result is now known as Brahmagupta's theorem, which was rediscovered in Europe by Snell in 1619 C.E.

(*v*) Brahmagupta expressed the rule to find the circumradius of a visamaquadrilateral in this verse as half the square root of the sum of the squares of opposite sides.

हृदयं विषमस्य भुजप्रतिभुज कृतियोगमूलार्धम्

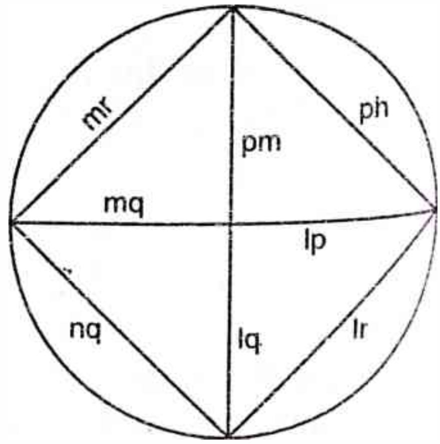
(Br. Sp. Si. XII 26)

(*vi*) The rule for the construction of rational cyclic quadrilateral is given by him in this sloka as

जात्यद्वयकोटिभुजाः परकर्णगुणाः भुजाश्चतुर्विषमे
अधिकोभूर्मुखं हीनो बाहुद्वितयं भुजावन्यौ॥

(Br. Sp. Si. XII 38)

According to this sloka, if $(l, m, n)$ and $(p, q, r)$ are two sets of sides of right angled triangled such that $l^2 + m^2 = n^2$ and $p^2 + q^2 = r^2$, then construct two triangles with sides $(lp, lq, lr)$ and $(pl, pm, pn)$ respectively, on the other sides construct the triangles with sides $(mp, mq, mr)$ and $(ql, qm, qn)$ respectively. These four triangles are shown in the adjacent figure, which form a quadrilateral with sides $nq$, $lr$, $np$ and $mr$ in order with its diagonals at right angles.




(*vii*) Brahmagupta also expressed the rule related to all those quadrilaterals which can be inscribed within circle and whose sides,

diagonals, areas, circumdiameters, segments, perpendiculars all expressible in integers. Such quadrilaterals are generally known as Brahmagupta quadrilateral. If sides of two right triangles are given by $p^2 - q^2, 2pq, p^2 + q^2$ and $r^2 - s^2, 2rs, r^2 + s^2$ then the sides of Brahmagupta quadrilateral are

$$(p^2 - q^2)(r^2 + s^2)\ ; \quad (r^2 - s^2)(p^2 + q^2)\ ;$$
$$2pq(r^2 + s^2) \quad \text{and} \quad 2rs(p^2 + q^2)$$

Brahmagupta also solved problems related to mensuration. He gave correct formula to find the volume of a pyramid in this sloka as

क्षेत्रफलं वेधगुणं समखातफलं ह्रतंत्रिभिः सूच्याः
मुखतल तुल्य भुजैक्यान्यैकाग्रहतानि समरज्जुः

(Br. Sp. Sid. XII 44)

From this sloka,

$$\text{the volume of a pyramid} = \frac{\text{Volume of a prism on the same base}}{3}$$

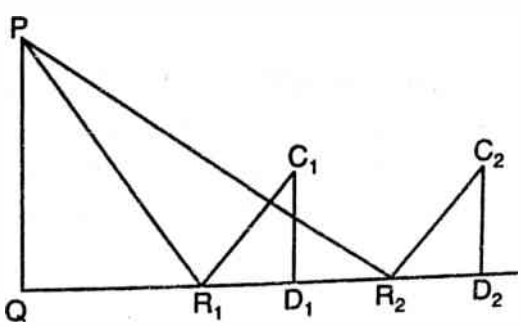
He also gave formula for the volume of a frustum first time as

$V = \frac{h}{3}(x^2 + y^2 + xy)$ ; where $x$ and $y$ are the sides of the base and the face and $h$ is the height. He explained the methods to calculate the time of the day from shadow measurements. In the chapter "Sankucchaya dijnanadhaya" dealing with the gnomon, shadow etc. He presented some interesting methods to determine the height and distance of objects by seeing their reflections in water. One of such methods is expressed in this sloka

प्रथम द्वितीय नृजलान्तरेणोद्धृता जलापसृतिः
दृष्टयौच्चयगुणोच्छायस्तोयान्नृ जलान्तरगुणा भूः॥

(Br. Sp. Si. XIX 19)

In figure, PQ is the house, $C_1D_1$ and $C_2D_2$ two positions of the observer and $R_1$, $R_2$ two positions of reflections, then




$$PQ = \frac{C_1D_1 \times R_1R_2}{D_2R_2 - D_1R_1}\ ;$$

$$QR_1 = \frac{D_1R_1 \times R_1R_2}{D_2R_2 - D_1R_1}$$

Brahmagupta also studied the topics related to trignometry, which he called as Jyotipattiganita *i.e.* the science of the calculation for the construction of sine. To find the values of R sine and R cosine of an arc, he gave the rule that these can be obtained from odd quadrants from the arc passed over and to be passed over. He formulated several trigonometric identities out of which few are listed below :

$$\text{R}-\text{utjya}\ \theta = \text{jya}\ (90-\theta)$$

$$\text{R}-\text{utjya}\ (90-\theta) = \text{jya}\ (\theta)$$

$$\sqrt{\text{R}^2-(\text{jay}\ \theta)^2} = \text{jya}\ (90-\theta)$$

$$\sqrt{\text{R}^2-\{\text{jya}\ (90-\theta)\}^2} = \text{jya}\ \theta$$

$$\text{R}+\text{jya}\ (\theta-90) = \text{utjya}\ \theta$$

$$\text{jya}\ \theta = \frac{\text{R}(180-\theta)\,\theta}{10125-(180-\theta)\left(\frac{\theta}{4}\right)}$$

He also gave inverse formula to find the arc approximately from corresponding R sine function as $\theta = 90 - \sqrt{8100 - \left(\frac{10125\,m}{(m/4+r)}\right)}$.

To find the radius of a circle circumscribed about a triangle Brahmagupta presented a correct trignometric result equivalent to

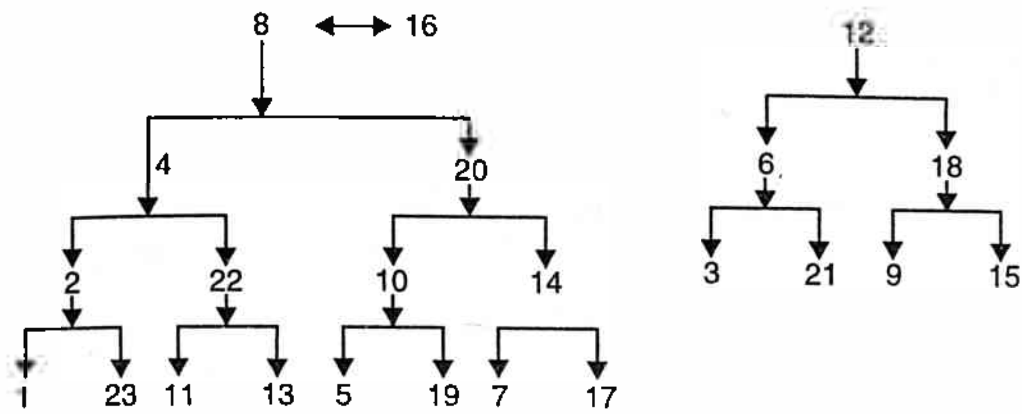
$$2\text{R} = \frac{a}{\sin \text{A}} = \frac{b}{\sin \text{B}} = \frac{c}{\sin \text{C}}$$

Brahmagupta applied the trigonometric identies

(*i*) $\sin\left(\frac{\theta}{2}\right) = \frac{1}{2}\sqrt{\sin^2\theta + \text{ver}\sin^2\theta}$

(*ii*) $\sin\left(90^\circ - \frac{\theta}{2}\right) = \sqrt{1-\sin^2\left(\frac{\theta}{2}\right)}$

to construct sine table. He called R sin $n\theta$ as the $n$th R sine, for which successive values can be obtained from following :




He constructed R sine-tables by taking radius 327° and angle interval 3°45′ for one table and radius 150 and angle intervel 15° for other.

Brahmagupta was the first mathematician who gave the second difference interpolation for calculation of R sine of any intermediate angle at equal interval of 15° of the arc as shown in following table :

| *Arc* | *jya (Indian sine)* | *First Difference* | *Second Difference* |
|---|---|---|---|
| 0° | 0 | | |
| | | 39 | |
| 15° | 39 | | – 3 |
| | | 36 | |
| 30° | 75 | | – 5 |
| | | 31 | |
| 45° | 106 | | – 7 |
| | | 24 | |
| 60° | 130 | | – 9 |
| | | 15 | |
| 75° | 145 | | |
| | | 5 | –10 |
| 90° | 150 | | |

To get interpolated values he expressed that "multiply half the difference of the gatakhanda (tabular difference to be passed over) by the residual arc and divide by 900 minutes. The result is to be added to or subtracted from half the sum of the same tabular differences according as this (semi sum) is less or greater than the bhogyakhanda, the final result is the true functional difference to be passed over". (Tr. by A.K. Bag)

which means,

$$p = \frac{1}{2}(p_i + p_{i+1}) \pm \frac{1}{2}(p_i - p_{i+1})\left(\frac{\alpha}{h}\right)$$

accordingly $p_i \leq p_{i+1}$ and

$$f(u + \alpha) = f(u) + \frac{\alpha}{h} \cdot p$$

By putting $\alpha = nh$, we get

$$f(u + nh) = f(u) + \frac{n}{2}[\Delta f(u - h) + \Delta f(u)] + \frac{n^2}{2}[\Delta f(u) - \Delta f(u - h)]$$

$$= f(u) + \frac{n}{2}[\Delta f(u - h) + \Delta f(u)] + \frac{n^2}{2}\Delta^2 f(u - h)$$

This rule of Brahmagupta is equivalent to the formula given by Newton and Stirling upto the second difference term, which was given by Brahmagupta almost one thousand years before the time it became known in Europe.

It is appropriate to mention here that the first two texts of Hindu mathematicians which were translated by Arabs were Brahma-sphuta-Siddhanta and Khandakhadyaka which they named as Sind Hind and Al-Arkand and through these texts only the treasure of mathematics discovered by Hindu mathematicians upto that time travelled different parts of Arab countries and thereafter made inroads to Europs and finally became known all over the world. Considering the valuable countribution of Brahmagupta Prof. Sachau expressed that "Brahmagupta occupies an important place in the history of oriental culture. Brahmagupta taught astronomy to the Arabs before they came to know of Ptolemy's works, since, references to the works, Sindhind and Al-Arkand frequently occur in Arabic literature ; these are the translations of Brahmagupta's works Brahma-Sphuta-Siddhanta and Khandakhadyaka". Bhaskaracharya II also had deep admiration for Brahmagupta and conferred him with the title "Ganaka-Cakra-Chudamani" *i.e.* the jewel in the galaxy of mathematicians.

# 9. Virasena

Virasena was a jaina scholar and mathematician of eighth century. He was contemporary of the Rastrakuta king Jagatungadeva. He wrote the Dhavala Tika of the Satkhandagma, which is considered a commentary on Jaina mathematics. Some Scholars are also of the opinion that mathematics contained in Dhavala Tika of Virasena is three-four centuries olders than his time. It seems that ancient Jaina mathematicians were familiar with the concept that there is a fixed ratio between circumference of any circle and its diameter. These scholars fixed different values for this ratio *i.e.* 3, $\sqrt{10}$ etc. and considered these values more or less as approximations. Virasena gave a better value for this fixed ratio of circumference and diameter. He stated that :

व्यासं षोडशगुणितं षोडशसहितं त्रिरूपरूपैर्भक्तम्
व्यासं त्रिगुणितं सूक्ष्मादपि तद्भवेत् सूक्ष्मम्

(Satkhandgama Vol. IV p.42)

This sloka gives the better value for the circumference of any circle by expressing that if diameter of any circle multiplied by 16, added with 16 and divided by 113 and further added with thrice the diameter, then the most exact value of circumference of circle can be obtained. Here, if circumference is denoted by C and diameter by $d$, then

$$C = \frac{16d + 16}{113} + 3d$$

From this expression if constant term 16 is taken away, then the value of C should be equal to $\frac{16d}{113} + 3d = \frac{355}{113}\,d$.

The value of this fixed ratio, which is denoted by $\pi$ is equal to

$$\left(\frac{C}{d}\right) \text{ or } \pi = \frac{\left(\frac{355}{113}d\right)}{d} = \frac{355}{113} = 3.1415929 \ldots$$

which is considered better approximation of $\pi$.

In the works of ancient Jaina scholars we find some references related to volumes and surface areas of solid figures. Virasena also dealt with geometry of various plans and solid figures and propounded the formulae to get the area, volume surface area and other related and relevant parameters of plane and solid geometrical figures. Virasena gave a beautiful geometrical demonstration of Karangathas, in which atoms or shots or very minute entities were arranged in the form of trapezoidal solid figure. He explained the rule to find the volume of such Karanagatha, *i.e.* trapezoidal solid in this verse as

मुहतलसमास-अद्धंवुस्सेधगुणं गुणं च वेधेण
घणगणिदं जाणिज्जो वेत्तसणसंठिए खेत्ते।

(Satkhandgama, part IV, p.20)

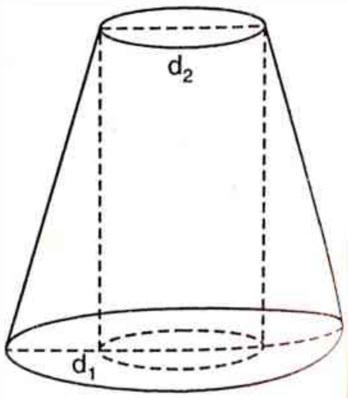
According to this verse the volume of this solid figure should be equal to the product of half of the sum of the face and the base ; the height and the depth *i.e.*

$$\text{Volume of trapezoidal solid} = \frac{1}{2}(\text{face} + \text{base}) \times \text{height} \times \text{depth}$$

He further expressed the rule to find the other form of Karanagatha, which is the combination of two trapezoidal solids joined together with their largest face and resemble like mrdanga in the following verse :

मूलं मज्झेण गुणं मुहसहिदद्धमुस्सेधकदिगुणिदं च
घणगुणिदं जाणिज्जो मुइंगसंठाणखेत्तम्मि।।

This verse gives the volume of this solid figure resembling like mrdanga as the product of the square of height and half of base multiplied by the middle and combined with the face. These formulae were also followed by other jaina scholars of later period. Virasena also gave the method of infinite division for finding the volume of a cone frustum. This method is given below.



If $d_1$ and $d_2$ are the diameters at the base and at the top and $h$ the height of the frustum, from the center of the frustum, as shown in figure, a cylindrical core of diameter $d_2$ is taken away and remaining sheath can give wedgeshaped solid, whose top edge is $\pi d_2$ and base is a trapezium with parallel sides $\pi d_1$ and $\pi d_2$ and height $\frac{d_1 - d_2}{2}$, when slit open vertically. A small wedge from its centre can be removed and its volume will be equal to $\pi d_2\left(\frac{d_1 - d_2}{2}\right)h$. After taking this small wedge away two wedges on a triangular base are arranged in such a way that they form rectangular parallelopiped of sides $\frac{h}{2}$ and $\frac{\pi(d_1 - d_2)}{4}$ and thickness $\frac{d_1 - d_2}{2}$. So its volume is $\pi\left(\frac{d_1 - d_2}{4}\right)\left(\frac{d_1 - d_2}{2}\right)\frac{h}{2} = \frac{\pi(d_1 - d_2)^2 h}{16}$

The remaining four triangular wedges are again cutup to form two parallelopipeds of volume

$$= \frac{2\pi(d_1 - d_2)}{8}\left(\frac{d_1 - d_2}{4}\right)\frac{h}{4} = \frac{\pi(d_1 - d_2)^2 h}{64}.$$

The remaining eight triangular wedges similarly produce four parallelopipeds of combined volume $\frac{\pi(d_1 - d_2)^2 h}{64 \,.\, 4}$.

This process is repeated infinitely till the remaining triangular wedges are infinitely small and the volume of the parallelopipeds formed by them can be neglected.

In this way, the total volume of all these parellopiped can be taken together as

$$\frac{\pi(d_1 - d_2)^2 h}{16}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \ldots\ldots\right)$$

$$= \frac{\pi(d_1 - d_2)^2}{16} h \left\{\frac{1}{1 - 1/4}\right\}$$

$$= \frac{4}{3}\frac{\pi(d_1 - d_2)^2 h}{16}$$

∴ The volume of the frustum

$$= \frac{\pi d_2^2}{4} h + \frac{\pi d_2 (d_1 - d_2) h}{4} + \frac{4}{3} \frac{\pi (d_1 - d_2)^2 h}{16}$$

$$= \frac{\pi h}{4}\left[d_2^2 + d_1 d_2 - d_2^2 + \left(\frac{d_1^2 + d_2^2 - 2d_1 d_2}{3}\right)\right]$$

$$= \frac{\pi h}{4}\left[\frac{3d_2^2 + 3d_1 d_2 - 3d_2^2 + d_1^2 + d_2^2 - 2d_1 d_2}{3}\right]$$

$$= \frac{\pi h}{4}\left[\frac{d_1^2 + d_1 d_2 + d_2^2}{3}\right]$$

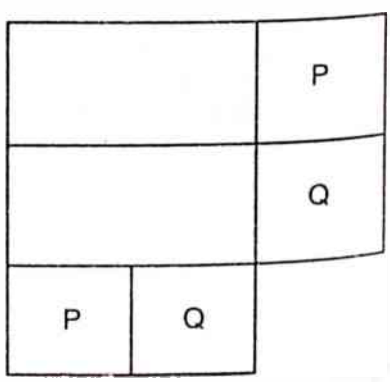
With this derivation it also becomes clear that Jaina mathematicians were quite comfortable in dealing with the problems related to the series and applied their expertise in solving several problems related to geometry and mensuration. Virasena also applied the concept of sum of infinite geometric series in getting the volume of frustum.

It seems that Jaina mathematicians were also demonstrating various algebraic identities in the form of geometry. Virasena also gave reference in Dhavala, a commentary on Satkhandagama related to geometrical representations of algebraic identities. For one algebraic identity he stated that

द्विभागाभ्यधिक सर्व जीव राशिना सर्वजीवराश्युपरिमवर्गे भागे हृते किमागच्छति ?
त्रिभाग हीन सर्व जीव राशिरा गच्छति, केन कारणेन् ......

(Satkhandgama Vol. III, p.44)

In this verse a question is asked that what should be the value if square of total number of creatures is divided by that total number increased by its half ? Again answer is also given in the form of equation only that result should be total number of creatures diminished by one-third of itself. Why ? Further it is explained geometrically that the figure which represents the square of the number of creatures should be divided into three parts by the lines drawn from east to west. One of these parts is to be divided into two equal parts and joined to the other two parts, the



width of those two parts, will be two third of the total number of creatures. Virasena called such geometrical representation by the name "bhagayamaksetra".

This method of geometrical demonstration can also be further extended to more parts like if a square of side $x$ is to be divided by $\left(1+\frac{1}{n}\right)$ of itself, then square should be divided into $(n + 1)$ equal parts, out of these $(n + 1)$ equal parts, one $(n + 1)$th part further should be divided into $n$ equal parts and these divided parts should be joined with other parts, forming a rectangle of length $x\left(1+\frac{1}{n}\right)$ and breadth $x\left(1-\frac{1}{1+n}\right)$. The required value is obtained from breadth.

It is interesting to note that Virasena also dealt with logarithms to base 2 (ardhaccheda) and he was the first to deal with logarithms to base 3 (trakacheda) and base 4 (caturthacheda).

# 10. *Lalla*

There are lot of uncertainties about the life period of Lalla. Pandit Sudhakar Dvivedi was of the opinion that Lalla might have lived around 499 A.D. but S.B. Dixit fixed it around 738 A.D. We also find a reference regarding the life period of Lalla in the "Consolidated List of Hindu Mathematical Works" given by K. Balagangadhran where he fixed it around 748 A.D. Lalla's father's name was Trivikram Bhatt and his grand father was Samba. Lalla followed the tradition of Aryabhata I, Brahmagupta and in general Aryabhata School of Kusumpura and contributed in the fields of Astronomy and Mathematics. His famous astronomical work is Sisyadhivrddhida which contains nearly one thousand slokas. In this text Lalla gave important results related to Trignometry. Because of his proximity with Aryabhata school he developed his work based on the Aryabhatia of Arybhata I and it had such a far reaching impact that even Bhaskaracharya (Bhaskara II) was also prompted to write a commentary on Sisyadhivrddhida, which he named as Vivarna and it seems that this commentary of Bhaskaracharya remained unpublished.

Bhaskaracharya also stated that Lalla authored one text related to mathematics with a title Pattiganita. To study the topics related to arithmetic and algebra Lalla wrote another text with the name Siddhantatilaka. His astronomical text Sisyadhivrddhida consists of two volumes. The first volume is spread into thirteen chapters and concentrated on the computation of the positions of the planets and covers the topics as mean longitudes of the planets, true longitudes of the planets, lunar and solar eclipses, syzygies, risings and settings, the three problems of diurnal rotation, the conjunction of the planet each other, the moons crescent, the shadow of moon, the patas of the moon and sun, the conjunction of the planets with fixed stars. The second volume was concentrated on the sphere in which the principle of mean motion, the terrestrial sphere, motion and stations of the planets, graphical representation, the celestial

shpere, geography etc. Lalla also wrote a commentary on Khandakhadyaka of Brahmagupta but this commentary could not be traced and is supposed to be lost. Lalla also wrote a text on astrology with a title Jyotisaratnakosha, which was very popular text and followed by the astrologers of later period for many centuries. Lalla was the first astronomer, who described Perpetuum Mobile in his text Sisyadhivrddhida. Lalla very frequently used the place value system in his texts with the help of Sanskrit numerical symbols. Ifrah said about the use of Sanskrit numerical symbols by Indian astronomer as" ..... over the centuries, Sanskrit has lent itself admirably to the rules of prosody and versification .... This explains why Indian Astronomers favoured to use of sanskrit numerical symbols, based on a complex symbolism, which was extraordinarily fertile and sophisticated possessing as it did an almost limitless choice of synonyms".

Indian astronomers used various astronomical instruments for measuring time, motion of plants and stars. These instruments were,

गोलो भगणश्चक्रं धनुर्घटी शङ्कुशकटकर्तर्यः
पीट्टक पालशलाका द्वादशयन्त्राणिसह यष्ट्या:

*i.e.* the Gola, the Bhagana, the Chakra, the Dhanus, the Ghati, the Shanku, the Shakata, the Kartari, the Pitha, the Kapala, the Shalaka, and the Yashti. Besides these twelve instruments Lalla also used water instruments, the self rotating globe and shadow instruments. Lalla for his astronomical calculations gave several formulae related to trigonometry. It is well known that, when a circle is divided by two perpendicular lines east to west line and north to south line it divides circle into four equal parts called Vrtta-pada, and these vrttapada are further classified as ayugma or visama (odd) and yugma or sama (even). Lalla expressed about these Vrtta-padas (quadrants), that these are formed by three anomalistic signs and they are successively called as odd and even. Regarding the functions of a complement or supplement he expressed that "when the anomaly is greater than 90°, it is subtracted from the semicircle, *i.e.* 180°, when greater than the semicircle, 180° is subtracted from it, when greater than 270°, it is subtracted from the complete circle *i.e.* 360°, the remainder is called the corresponding shuja".

(Tr. by Datta and Singh)

Lalla formulated following trigonometric identities as

$$\sqrt{R^2 - (\text{jya } \alpha)^2} = \text{kojya } \alpha$$

or $$\text{kojya } \alpha = \text{jya } (90 - \alpha)$$

Lalla constructed table of R sines and versed R sines for the radius 3438′. The method followed by the Lalla for the computation is almost similar to the method given by Aryabhata I and Surya Siddhanta. He also devised one shorter version of table for R sines and their differences by taking intervals of 10° of arcs of a circle of radius 150.

# 11. *Prthudakasvami*

The details about the life history of Prthudakasvami are not known but few informations which could be gathered from various sources made it clear that he wrote a valuable commentary on the Brahmasphutasiddhanta of Brahmagupta and explained various theorems, rules, principles propounded by Brahmagupta by illustrating them with the help of examples ; because in the original text of Brahmagupta no examples are found and those rules, theorems of Brahmagupta first time appeared with the examples in the text of Prthudaka svami. He described twenty logistics related to mathematics in his text, which he named as : 1. Samkalita (addition) 2. Vyavakalita (subtraction) 3. Gunana (multiplication) 4. Bhagahava (division) 5. Varga (square) 6. Vargamula (squre root) 7. Ghana (cube) 8. Ghanamula (cube root) 9-13. Pancagati 14. Trairasika (the rule of three) 15. Vyasta-Trairasika (the inverse rule of three) 16. Pancarasika (the rule of five) 17. Saptarasika (the rule of seven) 18. Nava-rasika (the rule of nine) 19. Ekadesarasika (the rule of eleven) 20. Bhanda-Pratibhanda (Barter and Exchange).

In his commentry we find the details about the fundamental operations of Hindu Ganita. He also stated that the method of multiplications like Kapatasandhi, Tatstha and Khanda, which were described by most of the Hindu mathematicians, should be attributed to the scholar of earlier period Skandasena, whose works could not be traced and this important information about Skandasena could only be obtained through the commentary of Prthudakasvami. He also treated various types of commercial problems related to Bater and Exchange, few of them are listed below for reference :

(*i*) A horse was purchased by (nine dealers in partnership, whose contributions were one etc. upto nine and was sold by them for five less than five hundred, find each man's share.

(*ii*) Four colleges, containing an equal number of pupils, were invited to partake of a sacrificial text. A fifth, a half, a third and a quarter came from the respective colleges to the feast and added to one, two, three and four, they were found to amount to eighty seven ; or with those deducted, they were sixty seven. Find the actual number of the pupils that came from each college.

(*iii*) Three jars of liquid butter, of water and of honey contained thirty two, sixty and twenty four pala respectively, the whole was mixed together and the jars filled again, find the quantity of butter, of water and of honey in each jar.

(*i* to *iii* translated by Datta & Singh)

Prthudakasvami used the term Kuttaka-Ganita for Algebra, where Kuttaka means pulveriser and used particularly in solving the equations of indeterminate nature. He named the coefficients of any unknown in the algebraic expression or equation by anka (number) or prakrti (multiplier). He also used colours to denote the unknown quantities like Kalaka (black), Nilaka (blue), Pitaka (yellow) and Haritaka (Green). It seems that Prthudakasvami was aware of the positive and negative roots of any quadratic equation and stated that while taking square root either of those roots can be considered according to the suitability of it for subsequent operations. He represented the equation in the following way :

*e.g.* $10x - 8 = x^2 + 1$

| | | | | | |
|---|---|---|---|---|---|
| yava | 0 | ya | 10 | ru | $\dot{8}$ |
| yava | 1 | ya | 0 | ru | 1 |

where ya is the abbreviation for yavat – tavat, which means the unknown quantity, ru is for rupa, which means constant term and yava is an abbreviation for yavatavadvarga, which means the square of the unknown quantity. With this system of representation the above equation can be put in the form

$0.x^2 + 10.x - 8 = x^2 + 0.x + 1$

Similarly, if any equation has several unknowns then that can also be expressed by this system of representation as shown below :

$197x - 1644\,y - z = 6302.$

This equation was represented by Prthndakasvami as

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ya | 197 | ka | $\dot{1644}$ | ni | $\dot{1}$ | ru | 0 |
| ya | 0 | ka | 0 | ni | 0 | ru | 6302 |

So, by putting $x$ for ya, $y$ for ka and $z$ for ni, we get

$$197x - 1644y - z + 0 = 0.x + 0.y + 0.z + 6302.$$

Prthudakasvami classified algebraic equations into four classes as (*i*) linear equations with one unknown (*ii*) linear equations with more unknowns (*iii*) Equations with one, two or more unknowns in their second and higher powers (*iv*) equations involving products of unknowns. He explained the solutions of the indeterminate equations of the form $by - ax = \pm c$, with the help of examples. He called the constant pulveriser of the form $by = ax \pm 1$ by Drdha-Kuttaka, where Drdha means firm. To present the solutions of simultaneous indeterminate equations like $N = a_1x_1 + r_1 = a_2x_2 + r_2$, he followed the method by reducing the two divisiors by reducing the two divisiors by common measure, as shown in following example : Find a number which leaves remainder 5, 4, 3, 2, when divided by 6, 5, 4, 3 respectively

$$N = 6x + 5 = 5y + 4$$

or $$x = \frac{5y - 1}{6}$$

to get the value of $x$ integral, $y$ and $x$ should be $6p + 5$ and $5p + 4$ respectively, so, $N = 30p + 29 = 4z + 3$

or $$p = \frac{2z - 13}{15}$$

To get $p$ integral, the values of $z$ and $p$ should be $z = 15q + 14$ and $p = 2q + 1$.

so, $$N = 60q + 59.$$

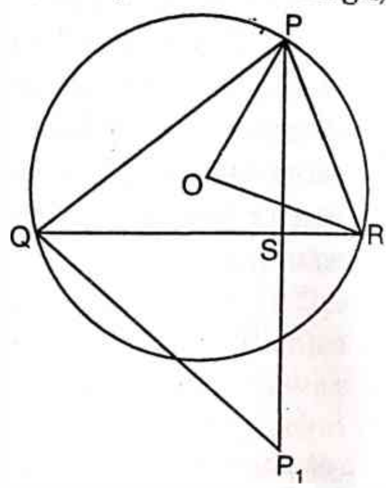
For the equations of the form $Nx^2 + 1 = y^2$ Prthudakasvami considered N as prakriti (multiplier), $x$ as kanistha-pada (lesser root) or adyamula, $y$ as jyestha-pada (greater root) or anyamula and C interpolator. He also gave solutions of square pulveriser like $bx + C = y^2$ and the equations of the form $axy = bx + cy + d$. He gave solutions of the equations like $5x - 25 = y^2$ ; $10x - 100 = y^2$ and $83x - 7635 = y^2$ as $y = 10$, $x = 125$ ; $y = 10$, $x = 20$ ; and $y = 1$, $x = 92$ respectively for these equations.

He also dealt with the problems related to geometry. He called the quadrilaterals like square, the rectangle and the isosceles trapezium by Avisama. According to G.R. Kaye, he was the first Indian mathematician who proved Ptolemys theorem, : "The sum of the products of the opposite sides of a cyclic quadrilateral remain equal to the product of its diagonals." He obtained the integral values of the sides of isosceles integral values of the sides of isosceles

triangle, scalene triangle and isosceles trapzium. He gave particular example by taking two right triangles with sides (3, 4, 5) and (5, 12, 13) and from these triangles he constructed a cyclic quadrilateral with sides (25, 39, 60, 52). Prthudakasvami gave the proof for the Brahmagupta's theorem related to circumcircle : "the product of the two sides of a triangle divided by twice the altitude is the heart line, twice of its is the diameter of the circle which passes through the corners of the triangles and quadrilaterals".

(Tr. by Datta & Singh)

**Proof :** Let PQR be scalene triangle, draw PS perpendicular to QR and produce it to P′ such that PS = SP′. Let O be the triangle of centre of the circle circumscribing the triangle PQR, join OP and OR, which make it clear that triangles QPP′ and OPR are similar so, PQ : OP : : PP′ : PR




or $\dfrac{PQ}{OP} = \dfrac{PP'}{PR}$ or $OP = \dfrac{PQ.PR}{PP'}$

where PQ and PR are the sides of triangle PQR, PP′ twice the altitude and OP, the radius of circumcircle.

He also illustrated few examples related to volume of pyramid and other geometrical figures.

# 12. *Mahaviracharya*

Mahaviracharya was an illustrious Jaina mathematician of 9th century A.D. We do not have sufficient authentic informations to detail his life history but it is almost certain that he flourished in the Court of Rastrakuta king Amogvarsa Nirputanga.(815 to 878 C.E.), who ruled over a large kingdom from a place Manyakheta (Malkhed) in Karnataka. Amogvarsha was the great patron of scholars and artists because he himself was a learned scholar who authored the texts Kavirajmarga in Kannad and Prasnottara Ratnamala in Sanskrit. Amogvarsa had very high respect for Jaina scholars Jinasena and Mahaviracharya. Mahaviracharya who had the patronage of king also called him reverentially as Cakrikabhanjan, Nirputang and Syadvadanyayavadi. Mahaviracharya had brillant capabilities blended together with intellectual astutism, poetic imagination with flavour of aesthetic articulations and creativity of an artist. Some historians are of the opinion that Mahaviracharya was the member of Jaina school of mathematics situated at Mysore and wrote his famous mathematical text "Ganita Sara Sangraha" around 850 C.E. in the language Sanskrit. This text was used as a textbook for a very long time in South India because of its lucid presentation, systematic developemenent of the topics, and compilation of the mathematical works done by his predecessors. In eleventh century "Ganita Sara Sangraha" was translated in Telugu by Pavuluri Mallana. It was the popularity of this text that large number of topics of Ganita Sara Sangraha were also found from Kerala which shows that this text was also read and referred in that region. During modern time this text came to light by the concerted efforts and researches of Prof. M. Rangacharya of Presidency College Madras, who discovered this text and got it published in 1912 by Madras Government. Mahaviracharya was the first mathematician of that period, who thought to author a text exclusively covering the topics related to mathematics. In Ganita Sara Sangraha he incorporated all those

rules, methods, theorems, principles etc. which were discovered by his predecessors. He elaborated those results and in some cases improved them and presented them in the form of a systematic compilation alongwith his own new results. Mahaviracharya held the position of mathematics in high esteem by expressing that

लौकिके वैदिके सामायिकेऽपि यः
व्यापारस्तत्र सर्वत्र संख्यानमुपयुज्यते॥
.........................................
.........................................
बहुभिर्विप्रलापैः किं त्रैलोक्ये सचराचरे।
यत्किंचिद्वस्तु तत्सर्वं गणितेन विना न हि॥

(G.S.S.I. 9, ... 16)

(in all transactions, which relate to worldly, Vedic or other similar religious affairs, calculation is of use ......, What is the good of saying much ? Whatever there is in all the three worlds, which are possessed of moving and nonmoving beings, can not exist without Ganita (Mathematics)) (Tr. by Datta and Singh)

He paid rich tributes to all those mathematicians whose works were included in "Ganita-Sara-Sangraha". He expressed that "with the help of the accomplished holy sages, who are worthy to be worshipped by the lords of the world and of their disciples and disciples' disciples, who constitute the well known series of preceptors, I glean from the great ocean of the knowledge of numbers a little of its essence, in the manner in which gems are picked from the sea, gold is from the stony rock and pearl from the oyster shell; and give out according to the power of my intelligence, the Sar-Sangraha, a small work on Ganita, which is (however) not small in value".

(Tr. by Datta and Singh)

Ganita Sara Sangraha comprises almost 1100 slokas covering various topics related to arithmetic, algebra geometry and mensuration detailed in nine chapters as given below.

- **Chapter 1.** Terminology (Sangyaadhikar).
- **Chapter 2.** Arithmetical Operations (Parikarm Vyvahar).
- **Chapter 3.** Operations on Fractions (Kalasvarna Vyvahar).
- **Chapter 4.** Miscellaneous Operations (Prakreenan Vyvahar).
- **Chapter 5.** Operations Involving the Rule of Three (Trairasik Vyvahar).

- **Chapter 6.** Mixed Operations (Misrak Vyvahar).
- **Chapter 7.** Calculation of Areas (Ksetraganita Vyvahar).
- **Chapter 8.** Operations relating to Excavations (Khat Vyvahar).
- **Chapter 9.** Operations Relating to Shadow Problems (Chhaya Vyvahar).

In this text, Mahavira made the classification of arithmetical operations and explained them by giving a number of examples. He covered almost all the topics related to basic arithmetical operations presented by his predecessors and represented them beautifully with the fragrance of his poetic flavour. He very frequently used the place value system with nine numerals and sign of zero alongwith Sanskrit numeration system and used the name for various numbers as Eka (1), Dasa (10), Shata ($10^2$), Sahasra ($10^3$), Dashasahasra ($10^4$), laksha ($10^5$), Dasalaksha ($10^6$), Koti ($10^7$), Daskoti ($10^8$), Shatakoti ($10^9$), Arbuda ($10^{10}$), Nyarbuda ($10^{11}$), Padma ($10^{14}$), Mahapadma ($10^{15}$), Kshoni ($10^{16}$), Mahakshoni ($10^{17}$), Shanka ($10^{18}$), Maha sanka ($10^{19}$), Kshiti ($10^{20}$), Mahakshiti ($10^{21}$), Kshoba ($10^{22}$), Mahakshobha ($10^{23}$).

Mahaviracharya expressed a particular number 1 2 3 4 5 6 5 4 3 2 1 by "ekadishadantani kramena hinani" which begins from 1 and increases by 1 successively until it reaches six and then decreases similarly in reverse order. This specific representation is sufficient to understand that he was familiar with place value system because without the knowledge of such system this representation is not possible. He also found some very peculiar numbers which remain same when read from left to right and vice-versa. These numbers also remain in the form of mala or necklace and because of such special characteristic, sometimes also named as garland products *e.g.*

$$139 \times 109 = 15151$$

$$12345679 \times 9 = 111\ 111\ 111$$

$$27994681 \times 441 = 12345654321$$

Mahavira gave four methods of multiplication namely, Kapta Sandhi, Tastha, Rupa-Vibhaga and Sthana-Vibaga. He also gave rules for division, square, cube and square root. Regarding square roots he clearly explained that :

धनं धनर्णयो वर्गोमूले स्वर्णे तयोः क्रमात्

ऋणं स्वरूपतोऽवर्गो यतस्तस्मान्न तत्पदम्॥

(Sangyadhikar 52)

According to it, the square of a positive and negative number is always positive but negative number can not have square root because negative number by its own nature is not a square of any other number. To find the cube of a natural number, he gave several methods which were based on the concept of progression. He made it clear that the sum of that arithmetic progession whose first term is $n$, common difference $2n$ and total terms in the progression again $n$, will be $n^3$. From this rule we can find following results quickly :

$$4 + 12 + 20 + 28 = 4^3 = 64$$
$$5 + 15 + 25 + 35 + 45 = 5^3 = 125$$
$$6 + 18 + 30 + 42 + 54 + 66 = 6^3 = 216$$

In general,

$$n + 3n + 5n + \ldots\ldots \text{ upto } n \text{ terms} = n^3$$

The other rules given by him to find the cubes are given below

(*i*) $(p + q)^3 = p^3 + 3p^2 q + 3pq^2 + q^3$

This rule can be generalised further as

$$(p + q + r + \ldots\ldots)^3 = p^3 + 3p^2(q + r + \ldots\ldots) + 3p(q + r + \ldots\ldots)^2 + (q + r + \ldots\ldots)^3$$

(*ii*) $n^3 = n(n + p)(n - p) + p^2(n - p) + p^3$

(*iii*) $n^3 = 3\sum_{p=2}^{n} p(p-1) + n$

(*iv*) $n^3 = n^2 + (n - 1)\{1 + 3 + 5 + \ldots\ldots + (2n - 1)\}$

He also generalised the summation of the square and the cube of a series of the form $p, p + q, p + 2q$ ..... as

$$p^2 + (p + q)^2 + (p + 2q)^2 + \ldots\ldots + \text{upto } n \text{ terms}$$

$$= n\left[\left\{\frac{(2n-1)q^2}{q} + pq\right\}(n-1) + p^2\right]$$

and, $p^3 + (p + q)^3 + (p + 2q)^3 + \ldots + \text{upto } n \text{ terms}$

$$= S^2 q + Sp(p - q) \text{ if } p > q$$
$$= S^2 q - Sp(p - q) \text{ if } p < q$$

where, $S = p + (p + q) + (p + 2q) + \ldots$ up to $n$ terms

Mahaviracharya also gave the sum of following series which is considered his remarkable constribution in this area,

$$p + (pq \pm r) + \{(pq \pm r)q \pm r\} + [\{(pq \pm r)q \pm r\}q \pm r] + \ldots\ldots \text{ upto } n \text{ terms}$$

$$= S' \pm \frac{\left(\frac{S'}{p} - n\right)r}{q-1}$$

where, $S' = p + pq + pq^2 + \ldots$ upto $n$ terms. He gave sum to the $n$ terms of following geometric progression,

$$p + pq + pq^2 + \ldots \text{ upto } n \text{ terms} = \frac{p(q^n - 1)}{q-1}$$

From this result, he deduced three more results to find the common ratio, first term and number of terms of any geometric progression.

The various forms of the fractions like $\left(\frac{p}{q} \pm \frac{r}{s}\right)$, $\left(\frac{p}{q} \text{ of } \frac{r}{s}\right)$, $\left(\frac{p}{q} \pm \frac{r}{s} \text{ of } \frac{p}{q}\right)$ or $\left(p \pm \frac{q}{r}\right)$, which are also named as jatis in Indian literature, were also explained by Mahavira. He enumerated six kinds of jatis namely

(*i*) Bhaga *i.e.*, the form $\left(\frac{p}{q} \pm \frac{r}{s} \pm \frac{t}{u} \pm \ldots\ldots\right)$

(*ii*) Prabhaga *i.e.*, the form $\left(\frac{p}{q} \text{ of } \frac{r}{s} \text{ of } \frac{t}{u} \ldots\ldots\right)$

(*iii*) Bhaganubandha *i.e.*, of the forms $\left(p + \frac{q}{r}\right)$

and $\left(\frac{p}{q} + \frac{r}{s} \text{ of } \frac{p}{q} + \frac{t}{u} \text{ of } \left(\frac{p}{q} + \frac{r}{s} \text{ of } \frac{p}{q}\right) + \ldots\ldots\right)$

(*iv*) Bhagapavaha *i.e.*, of the forms $\left(p - \frac{q}{r}\right)$

and $\left(\frac{p}{q} - \frac{r}{s} \text{ of } \frac{p}{q} - \frac{t}{u} \text{ of } \left(\frac{p}{q} - \frac{r}{s} \text{ of } \frac{p}{q}\right) \ldots\ldots\right)$.

(*v*) Bhaaga-bhaga *i.e.*, the forms

$\left(p \div \frac{q}{r}\right)$ or $\left(\frac{p}{q} \div \frac{r}{s}\right)$

(*vi*) Bhaga-matr *i.e.*, combinations of the forms shown above. Mahavira pointed out that twenty six variations of such combinations are possible by following the rule

$$^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 26$$

Mahavira was the first Hindu mathematician who equated the concept of nirudha with lowest common multiple *i.e.*, L.C.M. To find the sum of fractions of different denominators he expressed that

छेदापवर्तकानां लब्धानां चाहतौ निरुद्धः स्यात्
हरहृत निरुद्धगुणिते हाराशं गुणे समो हार:

(Kalasvarna Vyvahar

Mahavira's important contribution in the area of fraction is in devising the methods to express a fraction as the sum of several unit fractions. The type of unit fractions, which have unity in the numerator, was named by him 'rupam-saka-rasi'. The rules given by him about these unit fractions are given below

(*i*) $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\ldots + \frac{1}{3^{n-2}} + \frac{1}{2 \cdot 3^{n-2}}$

(*ii*) $1 = \dfrac{1}{2 \cdot 3 \cdot \frac{1}{2}} + \dfrac{1}{3 \cdot 4 \cdot \frac{1}{2}} + \ldots\ldots + \dfrac{1}{(2n-1)2n\left(\frac{1}{2}\right)} + \dfrac{1}{2n\left(\frac{1}{2}\right)}$

(*iii*) $\dfrac{1}{n} = \dfrac{a_1}{n(n+a_1)} + \dfrac{a_2}{(n+a_1)(n+a_1+a_2)} + \ldots\ldots$

$$+ \frac{a_{r-1}}{(n+a_1+a_2+\ldots+a_{r-2})(n+a_1+a_2+\ldots a_{r-1})}$$

$$+ \frac{a_r}{a_r(n+a_1+a_2+\ldots a_{r-1})}$$

By taking $a_1 = a_2 = \ldots. = a_r = 1$, we get unit fractions

(*iv*) $\dfrac{p}{q} = \dfrac{1}{r} + \dfrac{i}{r \cdot q}$ where $r = \dfrac{q+i}{p}$

(*v*) $\dfrac{1}{n} = \dfrac{1}{p \cdot n} + \dfrac{1}{\left(\frac{p \cdot n}{p-1}\right)}$

$$\frac{1}{a \cdot b} = \frac{1}{a(a+b)} + \frac{1}{b(a+b)}$$

(vi) $\frac{m}{n} = \frac{a}{\left(\frac{ap+b}{m}\right)\frac{n}{p}} + \frac{b}{\left(\frac{ap+b}{m}\right)\frac{n}{p} \times p}$

Mahavira gave solutions to many problems related to commercial mathematics by using the Rule of three and following relation

$$\frac{x_1}{c_1t_1} = \frac{x_2}{c_2t_2} = \frac{x_3}{c_3t_3} = \ldots = \frac{x_1 + x_2 + x_3 + \ldots\ldots}{c_1t_1 + c_2t_2 + c_3t_3 + \ldots\ldots}$$

which in turn also make clear that he was familiar with the algebraic identities like

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots = \frac{a+c+e+\ldots\ldots}{b+d+f+\ldots\ldots}$$

$$\frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d}$$

If we make a survey of literature related to Jaina school of thought, then it becomes interestingly clear that Jaina mathematicians had special fascination towards the topic prastarana and vikalpa which in modern notaion called as permutations and combinations. Mahaviracharya took a great leap forward in this directions by formulating ${}^nC_r = \frac{n\,!}{r\,!(n-r)!}$ or

${}^nC_r = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1, 2, 3, \ldots\ldots r}$. It is noteworthy to mention here that Mahavira was the first mathematician who introduced this formula, but unfortunately, some historians of the West wrongly attributed this discovery to Herigone (1634 CE), who rediscovered the same formula almost eight hundred years after Mahaviracharya.

Mahavira dealt with various topics related to algebra quite extensively. He gave methods to solve simultaneous linear equations and system of linear equations. A particular type of generalised system of linear equations of the form $b_1\,\Sigma x - c_1x_1 = a_1$ ; $b_2\,\Sigma x - c_2x_2 = a_2$ ..... $b_n\,\Sigma x - c_nx_n = a_n$ was also solved by him and he called such equations Citra-Kuttaka-Misra. He solved many problems related to interests by converting them into simple simultaneous equations involving several unknowns. While dealing with quadratic equations he considered both the roots *e.g.* to solve a quadratic equation of the form $\frac{a}{b}x^2 - x + c = 0$, he gave values of $x$ equal

to $\dfrac{b/a + \sqrt{(b/a - 4c)b/a}}{2}$. He also studied equations of higher degrees like $ax^n = q$ ; $a\left(\dfrac{x^n - 1}{x - 1}\right) = b$. For simultaneous quadratic equations like $x + y = a$ and $xy = b$, he obtained the values as $x = \frac{1}{2}(a + \sqrt{a^2 - 4b})$ and $y = \frac{1}{2}(a - \sqrt{a^2 - 4b})$. Similarly, for the equations $x^2 + y^2 = c$ and $xy = b$ he found value of $x = \frac{1}{2}(\sqrt{c + 2b} + \sqrt{c - 2b})$ and $y = \frac{1}{2}(\sqrt{c + 2b} - \sqrt{c - 2b})$.

Mahavira called the equations of indeterminate nature by "Kuttakara" and solved many problems related to indeterminate linear equations of first degree and simultaneous indeterminate linear equations of first degree. An interesting problem related to it is described here : "three merchants find a purse lying on the road, one merchant says ; if I keep the purse I shall have twice as much money as the two of you have together, second merchant says give me the purse I shall have thrice as much money as you two have, third merchant says if I keep the purse I shall have five times as much money as you two have together. How much money is in the purse ? How much money does each merchant have ?

If first merchant has $x$, the second $y$, the third $z$ and money in the purse is M, then $x + M = 2(y + z)$ ; $y + M = 3(x + z)$ ; $z + M = 5(x + y)$, for which there is no unique solutien but the smallest values for these are $x = 1, y = 3, z = 5$ and $M = 15$. Mahavira also stated that other solutions of this problem in positive integers should be a multiple of this solution.

Mahaviracharya made properties of geometrical figures clear by explaining them with suitable examples. He classified triangles into three kinds namely sama (equilateral) ; dvisama (isosceles) and visama (scalene). He was the first mathematician, who gave the rule to find the radius of a circle inscribed in a triangle (or quadrilateral if possible), whose area and perimeter is known. He expressed it in this sloka ;

परिधेः पादेन भजेदवलब्धक्षेत्र सूक्ष्मगणितं तत्
क्षेत्राभ्यन्तरवृत्ते विषकम्भोऽयं विनिर्दिष्ट

(G.S.S. VII 223)

From it, we get the value of radius of circle inscribed within a triangle of sides $a$, $b$ and $c$ as $r = \frac{2\Delta}{s}$, where $\Delta$ is area of triangle and $s$, semiperimeter of it.

Mahavira referred rational figures by Janya and dealt with all kinds of such figures under the topics Janyavyavahara. For rational right triangle he stated that

वर्गविशेषः कोटिः संवर्गो द्विगुणितो भवेद् बाहुः
वर्ग समासः कर्णश्चायतचतुरश्रजन्यस्य॥

(G.S.S. VII 90)

According to this sloka, the diagonal of generated rectangle should be sum of the squares of the upright and base where difference of the squares of two elements is the upright and twice their product is the base *i.e.* for the elements $m$, $n$ the upright base and diagonal of rectangle should be $m^2 - n^2$, $2mn$, $m^2 + n^2$. He explicitly dealt with the different cases of rational figures

($i$) for given hypotenuse $h$, the first solution *i.e.* the sides of rational right triangle should be $x^2$, $\sqrt{h^2 - x^4}$, $h$ for any rational number $x$.

($ii$) the second solution should be $x$, $\sqrt{h^2 - x^4}$, $h$

($iii$) the third solution should be equal to

$$\frac{m^2 - n^2}{m^2 + n^2}\,h, \left(\frac{2mn}{m^2 + n^2}\right)h, h$$

By applying this third rule Mahavira himself found four rectangles for the same diagonal 65 as (39, 52) ; (25, 60) ; (33, 56) and (16, 63). He also gave the rule to find rational rectangle (or square) whose area will be numerically equal to any multiple or submultiple of a side, diagonal or perimeter or of any linear combination of two or more of them. This can be expressed as

$$x^2 + y^2 = z^2$$
$$ax + by + cz = dxy,$$

for any rational numbers $a, b, c, d$ ; $d \neq 0$

Mahavira gave the rule to get an isosceles triangle from a single generated rectangle, for which the two sides of triangle, twice its side is the base, the upright is the altitude and the area of rectangle is equal to the area of isosceles triangle. So for any two integers $p$

and $q$ $(p \neq q)$ ; the sides of all rational isosceles triangles with integral sides are

(i) $p^2 + q^2$ ; $p^2 + q^2$ ; $2(p^2 - q^2)$

(ii) $p^2 + q^2$ ; $p^2 + q^2$ ; $4pq$

He also stated that if $(P_1, P_2)$ and $(A_1, A_2)$ denote the perimeters and areas of two rational isosceles triangles such that $P_1 : P_2 = m : n$ and $A_1 : A_2 = r : s$ where $m, n, r$ and $s$ are known integers, then triangles can be obtained by two generated rectangles with sides

$$\left(\frac{6n^2r}{m^2s}, \frac{2n^2r}{m^2s} - 1\right) \quad \text{and} \quad \left(\frac{4n^2r}{m^2s} + 1, \frac{4n^2r}{m^2s} - 2\right)$$

where $n^2r > m^2s$

The isosceles triangle from the sides of first rectangle has the dimensons

$$\text{side} = m\left\{\left(\frac{6n^2r}{m^2s}\right)^2 + \left(\frac{2n^2r}{m^2s} - 1\right)^2\right\}$$

$$\text{base} = 24m\left(\frac{n^2r}{m^2s}\right)\left(\frac{2n^2r}{m^2s} - 1\right)$$

$$\text{altitude} = m\left\{\left(\frac{6n^2r}{m^2s}\right)^2 - \left(\frac{2n^2r}{m^2s} - 1\right)^2\right\}$$

and from the sides of second rectangle, the dimensons of isosceles triangle are

$$\text{side} = n\left\{\left(\frac{4n^2r}{m^2s} + 1\right)^2 + \left(\frac{4n^2r}{m^2s} - 2\right)^2\right\}$$

$$\text{base} = 4n\left(\frac{4n^2r}{m^2s} + 1\right) + \left(\frac{4n^2r}{m^2s} - 2\right)$$

$$\text{altitude} = n\left\{\left(\frac{4n^2r}{m^2s} + 1\right)^2 - \left(\frac{4n^2r}{m^2s} - 2\right)^2\right\}$$

It is interesting to note here that Mahavira gave these results almost seven hundred years before its rediscovery in the West by

Younger in 1657 CE and Rahn in 1697 CE. He also treated rational scalene triangles and described the rules to find its sides, altitude and area.

Mahavira listed five kinds of quadrilaterals as Sama (square, rhombus); Dvidvisama (rectangle); Dvisama (isosceles trapazium), Trisama (trapezium with three sides equal and Visama (unequal sides). He was familiar about the formula $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$ for the area of a cyclic quadrilateral of sides $a$, $b$, $c$ and $d$ and also made it clear that this formula is not applicable to the visama-caturasara. He gave the rule or getting the face, base and equal sides of an isosceles trapezium which have area and altitude equal to another isosceles trapezium, whose dimensions are known. He further elaborated that if a rectangle generated from the rational sides $(a, b)$ be smaller than that of other rectangle generated from the rational sides $(l, m)$, then a rational inscribed quadrilateral can be obtained whose sides are

$(a^2 - b^2)(l^2 + m^2)(a^2 + b^2)$ ; $(l^2 - m^2)(a^2 + b^2)^2$ ; $2ab(l^2 + m^2)(a^2 + b^2)$ ; $2lm\,(a^2 + b^2)^2$ and area equal to

$$\frac{1}{2}\{2lm(a^2 - b^2) + 2ab\,(l^2 - m^2)\} \times \{(a^2 - b^2)(l^2 - m^2) + 4ab\,lm\}(a^2 + b^2)^2$$

While dealing with circle, Mahaviracharya gave the formula for the the area of an outlying and inlying annulus as $A = (d \pm p)\,\pi\,p$ ; where $d$ is the inner or outer diameter and $p$ the width of the annulus. He cosidered the value of $\pi$ approximately equal to 3. He gave accurate formula for the area of regions enclosed by the arcs of four mutually touching circles as the difference of the area of the square formed joining the centres of the four circles, with these centres as the vertices of the square and area of any one circle. Similarly, he explained that when three circles mutually touch each other, then the area of region bounded by the arcs of these circles should be equal to the difference of the area of equilateral triangle formed by the centres of these circles with vertices of triangle at these centres and half the area of any one circle. The remarkable contribution of Mahavira in the field of geometry and measuration is related to ellipse, which he called as Ayata-Vrtta. Though his formula for computation of the area of ellipse is wrong but his rule for the circumference of ellipse equal to $\sqrt{16a^2 + 24b^2}$, for longer diameter $2a$ and shorter diameter $2b$, reduces to the form

$2\pi a\sqrt{1-\frac{3}{5}e^2}$, which is very close approximation of the circumference, where the eccentricity $e = \sqrt{\frac{(a^2-b^2)}{a^2}}$.

Mahavira expounded an interesting rule to find the side of any regular polygon inscribed in a circle of given diameter in the sloka

लब्धव्यासेनेष्ट व्यासौ वृत्तस्य तस्य भक्तश्च
लब्धेन भुजा गुणयेत् भवेच्च जात्स्य भुजसंख्या

(G.S.S. VII 221)

According to it, to find the side of regular polygon the diameter of circle should be divided by the circumdiameter of any given polygon and further this ratio is to be multiplied with the sides of regular polygon (given).

He also gave several examples related to inverted cases to truncated pyramids on square, rectangular or equilateral triangular bases and truncated cones. He formulated the rule to find the volume of tetrahedron in this sloka.

भुजाकृत दलघनगुण दशपदनवहृत व्यावहारिकं गणितम्
त्रिगुणं दशपदभक्तं शृंगाटके सूक्ष्म घन गणितम्

(G.S.S. VIII 30)

In this sloka approximate volume of tetrahedron $V_a = \frac{\sqrt{10}}{18\sqrt{2}}a^3$, where $a$ is the edge of tetrahedron and exact volume

$$= V_a\left(\frac{3}{\sqrt{10}}\right) = \frac{\sqrt{10}}{18\sqrt{2}}a^3 = \left(\frac{3}{\sqrt{10}}\right) = \frac{a^3}{6\sqrt{2}}.$$

# 13. *Sridhara*

Sridharacharya, commonly named as Sridhara was a versatile mathematician, who wrote two mathematical texts Trisatika and Patiganita. Regarding life history of Sridhara nothing definite can be said because different scholars have putforth their own estimation to determine the period of his life and place of his birth. On the basis of these estimations ranging from seventh century A.D. to eleventh century A.D., most of the scholars are of opinion that most probably Sridhara lived during nineth and tenth centuries A.D. K.S. Shukla estimates his period of life between 850 and 950 A.D. and it is also supposed that Sridhara wrote his mathematical texts around 900 A.D. Again, for his place of birth also there are differences among the historians ; few of them are of the opinion that he was from Bengal region, whereas some others say that his place of birth was in Southern part of India.

Initially, it was believed that Sridhara wrote only two texts Trisatika and Patiganita but during later period many references were found in the works of Bhaskara II, Makkibhatta and Raghavabhatta, due to which few more texts namely Bijaganita, Navasati and Brahatpati were also attributed to Sridhara. Sridhara gave the names of places by using the terms "dasagunah samjnah" in Trisatika as Eka (1), Dasa (10), Sata ($10^2$); Sahasra ($10^3$), Ayuta ($10^4$), Laksa ($10^5$), Prayuta ($10^6$), Koti ($10^7$), Arbuda ($10^8$), Abja ($10^9$), Kharva ($10^{10}$), Nikharva ($10^{11}$), Mahasaroja ($10^{12}$), Sarita ($10^{13}$), Sarita-Pati ($10^{14}$), Antya ($10^{15}$), Madhya ($10^{16}$), Parardha ($10^{17}$). He also mentioned that this list of numbers can proceed further by following the rule of dasagunah at every stage. The Patiganita of Sridhara is written in the form of verses and covers the topics related to basic arithmetical operations, square, square root, cube, cube root etc. Sridhara gave four methods of multiplication namely Kapata-sandhi, Tastha, Rupa-Vibaga and Sthana-Vibaga. He formulated the rule to find the sum of series like

$$a^3 = \Sigma\{3a\,(a-1)+1\}$$

Shridara dealt with the problems of fractions (jati) and considered their six cases like :

(*i*) Bhaga *i.e.* $\left(\frac{l}{m} \pm \frac{n}{0} \pm \frac{p}{q} \pm \ldots\right)$

(*ii*) Prabhaga *i.e.* $\left(\frac{l}{m} \text{ of } \frac{n}{0} \text{ of } \frac{p}{q} \ldots\right)$

(*iii*) Bhaganubandha *i.e.* either $\left(l + \frac{m}{n}\right)$

or $\left(\frac{l}{m} + \frac{n}{0} \text{ of } \frac{l}{m} + \frac{p}{q} \text{ of } \left(\frac{l}{m} + \frac{n}{0} \text{ of } \frac{l}{m}\right) + \ldots\ldots\right)$

(*iv*) Bhagapavaha *i.e.* $\left(l - \frac{m}{n}\right)$

or $\left(\frac{l}{m} - \frac{n}{0} \text{ of } \frac{l}{m} - \frac{p}{q} \text{ of } \left(\frac{l}{m} - \frac{n}{0} \text{ of } \frac{l}{m}\right) + \ldots\ldots\right)$

(*v*) Bhaga-bhaga *i.e.* $\left(l - \frac{m}{n}\right)$ or $\left(\frac{l}{m} + \frac{n}{0}\right)$

(*vi*) Bhagamatr.

Sridhara also explained the Rule of three as " of the three quantities, the pramana and iccha, which are of same denominations should be placed at first and last and the phala should remain at the middle, the product of this and the last is to be divided by the first". He also solved the problems related to ratio, simple interest, purchase and sales etc. He formulated the rule to find the combinatorial value, when out of $n$ things, $m$ things are taken at a time. He solved the problems related to arithmetic and geometric progression and diagrammatically represented arithmetic series in the form of Srediksetras. He used trapezium to represent arithmetic progressions and also considered trapezium with negative bases or faces, in which flanks cross over each other. For any arithmatic progression Sridhara used the word adih for first term, gacchah for total number of terms, ganitam for sum of whole series and uttarah for common difference.

Sridhara gave the methods to find rational solutions for the equations like $Nx^2 \pm 1 = y^2$ ; $1 - Nx^2 = y^2$ ; $Nx^2 \pm c = y^2$ and $c - Nx^2 = y^2$. Sridhara is also credited to be the first scholar of mathematics who gave the systematic and comprehensive rule to find the solutions of a quadratic equation. Though the portions of his mathematical

text containing these solutions were lost, which later on came to the light, when Bhaskaracharya II made the reference of this method in his text and attributed it to Sridhara. Bhaskara II stated this method as "to find the solution of quadratic equation, multiply both the sides of given equation by a known quantity equal to four times the coefficient of the unknown ; add to both sides a known quantity equal to the square of the coefficient of the unknown ; then take the square root".

For any quadratic equation $px^2 + qx = r$, the steps of solution are

$$4p^2x^2 + 4pqx = 4pr$$

or
$$4p^2x^2 + 4pqx + q^2 = 4pr + q^2$$

or
$$(2px + q)^2 = 4pr + q^2$$

taking the square root we get

$$2px + q = \sqrt{4pr + q^2}$$

As already mentioned earlier, the original text related to this method is not available, so it could not be seen whether Sridhara considered both positive and negative values while taking the square root or considered only positive value, simply ignoring the negative value. The application of this rule is found in his text to find the number of terms of an arithmetic progression whose first term is $a$, common difference $b$ and sum of $n$ terms, then

$$n = \frac{\sqrt{8bs + (2a - b)^2} - 2a + b}{2b}$$

Sriddhara also gave the rules for calculating the time of the day from the length of shadow and vice-versa. For the area of quadrilateral Sridhava gave the formula as $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ without knowing the limitations of this formula with different type of quadrilaterals. He considered triangle as particular case of the quadrilateral and gave the formula for the area of triangle as $\sqrt{s(s-a)(s-b)(s-c)}$ or half of the product of its base and altitude. Sridhara also dealt with the geometry of circle and gave beautiful approximation for finding the area of a segment in this sloka.

जीवाशरैक्य दलहतशरस्य वर्ग दशाहतं नवभिः
विभजेदवाप्तमूलं प्रजायते कार्मुकस्य फलम्॥

(T.S. 47)

(the square of the arrow as multiplied by half the sum of the chord and the arrow should be multiplied by 10 and divided by 9. The square root of the quotient gives the area of the segments)

(Tr. by T.S. Amma)

area of segment = $\dfrac{\pi h(c+h)}{3 \times 2}$ ;

where $\pi = \sqrt{10}$

Sridhara also dealt with the geometry of solid figures. He gave the correct formula for the volume of a cone made by the heap of circular base as $\dfrac{\pi d^2 h}{12}$. For the volume of conical frustum he expressed that :

मुखतलतद्योगानां वर्गैक्यकृतेः पदं दशगुणायाः
वेधगुणं चतुरन्वितविंशति-भक्तं फलं कूपे:

(T.S. 38)

(the square root of ten times the square of the sum of the squares of the diameters at the top, the diameters at the top, the diameter at the bottom and the sum of these diameters, when multiplied by the depth and divided by 24, gives the volume of a well)

(Tr. by T.S. Amna)

*i.e.*, $$V = \frac{\pi}{24} h(d_1^2 + d_2^2 + (d_1 + d_2)^2)$$

where, $h$ is the height and $d_1$ and $d_2$ are the diameters of the base and the top.

# 14. Aryabhata II

Year of birth around 920 A.D. (Probable)
Year of death around 1000 A.D. (Probable)

The life history of Aryabhata II is full of disputes. Regarding his year of birth and year of death, scholars have drawn their own inferences about these years, on the basis of available literature of that period and commentaries written later on containing the references of his work. But most of the scholars are of the opinion that he lived probably in between 920 A.D. to 1000 A.D. and wrote his celebrated work "Mahasiddhanta" probably around 950 A.D. Mahasiddhanta of Aryabhata II contains eighteen chapters and this text is exclusively written in the form of Sanskrit verses. First twelve chapters of this text are devoted to mathematical astronomy covering the topics like : the longitudes of the planets, eclipses of the sun and the moon, the lunar crescent conjunctions of the planets with each other and with the stars. The remaining six chapters are devoted to Geography, Geometry and Algebra. To represent the numbers he made some modifications in Katapayadi system by considering vowels either themselves or in conjunction with constants without any numerical significance, wherever constants were given same value as in Katapayadi system. *e.g.* Aryabhata II denoted 488108674 in the form of letter alphabets as dha-jha-he-ku-na-he-t-sa-bha, which gives the value 47801884 in Katapayadi system. Aryabhata II explained all basic operations of arithmetic with more clarity and precision. As for additions, he stated that it is making into one of several numbers and subtraction, a process of taking out from the sarvadhana (total). He explained the ways of multiplication, division, squares, cubes etc. and also defined the rule of three, five, seven, nine or more terms. He also presented the methods to solve the problems related to progressions *e.g.* to find number of terms of an arithmetic progression, whose first term $a$, common

difference $d$ and sum to $n$ terms of the progression are known, he gave the formula

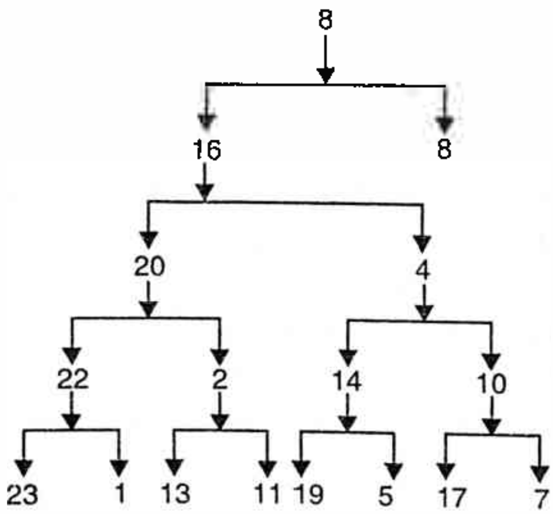
$$\text{number of terms} = \frac{\sqrt{2ds + \left(a - \frac{d}{2}\right)^2} - a + \frac{d}{2}}{d}$$

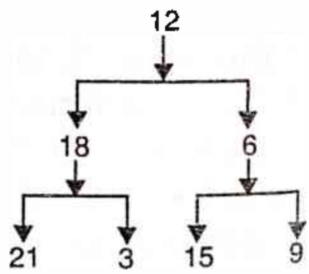
Aryabhata II also improved the methods given by his predecessors to solve the indeterminate equations of first degree like $ax - by = \pm c$, for positive integers. He also solved simultaneous indeterminate equations of first degree with a common divisor ; generally known as samslistakuttaka or the conjunct pulveriser by stating that dividend will be the sum of the multipliers and the interpolator the sum of the given interpolators.

Aryabhata II constructed sine table correct upto five decimal places. He also considered the decimal parts of radius which he found useful in calculating the positions of planets as accurately as possible. He was the first mathematician who formulated the trignometric identity

$$\sin\left(\frac{90° \pm x}{2}\right) = \sqrt{\frac{1}{2}(1 \pm \sin\theta)}$$

He took recourse of this formula in the construction of sine table, beginning with the known value of R sin 30° and R sin 45° and exhibited the table in the form




In the process of constructing R sine tables Aryabhata II took the radius 34 38′ and interval 225′. It is interesting to note that Aryabhata II also dealt with the problems related to differentials

in his text Mahasiddhanta, where we find the reference of the formula for the calculation of true motion in the form

$$u' - u = v' - v \pm e(\omega' - \omega) \cos \omega,$$

which in the terms of derivatives can be interpreted as

$$\delta u = \delta v \pm e \, \delta\omega \cos \omega.$$

Aryabhata II gave the approximate value of $\pi$ equal to $\frac{22}{7}$ while defining the rule to find the area and circumference of a circle. He explained that if diameter of a circle is multiplied by 22 and divided by 7 then it will give approximate value of the circumference and if square of semidiameter also multiplied by 22 and divided by 7, the approximate value of area of a circle can be obtained. It is particularly relevant to mention here that this approximate value of $\pi$ equal to $\frac{22}{7}$ appeared first time in Aryabhata II's Mahasiddhanta. He also gave another method to compute the value of $\pi$ by dividing circumference of a circle by its diameter. He stated that if circumference of a circle is 21600, multiply it by 191 and divide by 600, the quotient thus obtained give the value of diameter, which in turn gives the value of $\pi$ equal to $\frac{600}{191} = 3.14136.....$

Aryabhata II also dealt with the geometry and gave several methods to construct different types of plane geometrical figures. He was aware of the formulae : area of a triangle = $\frac{1}{2}$(base × altitude) or area of a triangle = $\sqrt{s(s-a)(s-b)(s-c)}$ to find the exact value of the area of a triangle. He was also familiar with the properties of triangle like : in any triangle no side can be greater than the sum of remaining sides and for some triangles the altitude drawn from any vertex may fall outside the triangle. Aryabhata II devised a rule to find the approximate area of a quadrilateral in this verse :

त्रिभुज चतुर्भुज फलानयनाय
वदनक्षितियोगदलं लम्बहतं जायते गणितम्
त्रिभुजे समचतुरस्रेऽर्धसमे वा कर्णभेदेऽपि
शृंगाटके न नियमाद्विषम चतुर्बाहुके च न प्रायः
याम्योत्तर लम्बैक्यार्द्ध वधा स्यैक्यार्धताडितं निकटम्

(Ma Si XIV. 78-79)

From it, the area of a triangle can be computed by multiplying half the sum of the face and base by the altitude, whereas the

approximate area of quadrilateral should be found by multiplying half the sum of the base and face to half the sum of opposite altitudes towards south and north. He also gave the method to calculate the diagonals of a quadrilateral. To find the area of square and rhombus he beautifully explained the rule in this verse :

श्रुतिघातः समचतुरस्रे अर्धितः फलं स्यात्

(Ma Si, XIV, 82)

which shows that if the product of diagonals of square or rhombus is halved then we get the area of these figures. He also explained the method to find the value of second diagonal of parallelogram or rhombus by establishing an identity in this verse :

समचतुरस्रेऽर्धसमे वाभीष्ट श्रवणवर्गोनात्

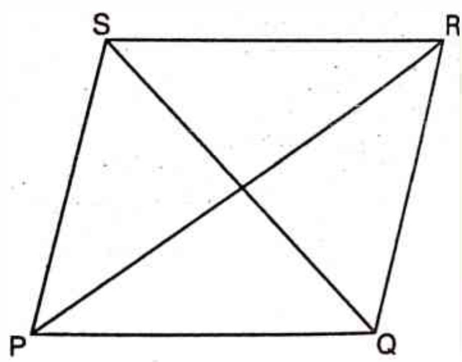
सर्वभुजवर्गयोगमूलं कर्णो द्वितीयः स्यात्

(Ma Si XIV 81-82)

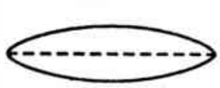
From this verse for any parallelogram PQRS we can get the identity

$$QS^2 = PQ^2 + QR^2 + SR^2 + SP^2 - PR^2$$

or $$QS^2 = 2PQ^2 + 2QR^2 - PR^2$$

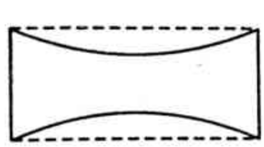


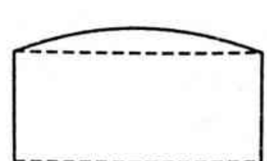

Aryabhata II also gave rule to find the approximate area of segment of a circle and explained many problems related to mensuration like barely corn, crescent moon, elephant's tusk, thunderbolt drum etc. These have the shapes as shown in following figures :


Barely corn


Crescent moon


Elephant's tusk


Thunder bolt


Drum

Aryabhata II explained crescent moon as a combination of two triangles, elephant tusk consisting of only one triangle, a barely corn consisting of two segments of a circle or two triangles, drum the combination of two segments of a circle outside and a rectangle inside and thunderbolt consisting of two segments of two circles and two quadrilaterals. He also gave formula for the volume of a cone or pyramid equal to the half product of area of its base and height. He also devised a formula related to excavations of various forms, which may be inverted truncated pyramids on square, rectangular or equilateral triangular bases. He stated that to find the volume of such figures, first divide the sum of the areas of the face and base and that arising from the sum of their sides by six and then multiply this value with the height. It can also be precisely formulated in following way by denoting the sides of the face by $p$ and $q$ and that of the base by $r$ and $s$ with height $h$, then volume should be equal to $\frac{h}{6}[(p+r)(q+s)+pq+rs]$.

Aryabhata II was also aware of the volume and surface area of sphere. To find the volume of any sphere he gave the rule that if cube of the diameter of a sphere is halved and then added to its eighteenth part, then we get approximate value of the volume of a sphere. For the surface area it seems that he used the formula *i.e.* surface area of a sphere equal to product of circumference and diameter.

# 15. Sripati

Sripati was a famous mathematician of 11th century. His father was Nagdeva, who was also called sometimes as Namadeva. He was born probably in the year 1019 A.D. in Rohinikhanda, a place situated in Maharashtra state of India. Sripati authored the texts related to astrology, astronomy and mathematics. In 1039 A.D., he wrote the book Dhikotidakarana, which covered the topics related to solar and lunar eclipses. He wrote another book named as "Dhruvamanasa" around 1056 A.D., containing 105 verses on the rules for calculation of planetary longitudes, eclipses and planetary transits. His major contribution in the area of astronomy was complied in the text, "Siddhantasekhara", which has 19 chapters. Ganitatilaka, a treatise on mathematics, he wrote to cover the topics related to arithmatic. Ganitatilaka contains 125 verses, and this text was based on the work of Sridhara. It seems that this text of Sripati is incomplete and few portions of it are missing. In this text he explained elaborately the laws of signs like :

(*i*) sum of two positives is positives ; of two negatives is negative ; of a positive and a negative is their difference and sign of the difference is that of greater ; of two equal positives and negatives is zero

(*ii*) on multiplying two negatives or two positives, product is positive ; in the multiplication of positive and negatives, the result is negatives

(*iii*) the square of a positive and a negative, result is positive ; a negative number itself is non square, so its square root is unreal. [(*i*) to (*iii*) translated by Datta and Singh]

Sripati also gave the rule to find square, square root, cube, cube root etc. of positive and negative quantities. He formulated the rule for solving the quadratic equations and established the identity

$$\sqrt{x+\sqrt{y}} = \frac{1}{2}\sqrt{(x+\sqrt{x^2-y})} + \frac{1}{2}\sqrt{(x-\sqrt{x^2-y})}$$

Sripati also devised the rules to find the solutions of linear equations in one variable and simultaneous indeterminate equations in one variable and dealt with the simultaneous indeterminate equations of the type

$$\begin{gathered} b_1 y_1 = a_1 x \pm c_1 \\ b_2 y_2 = a_2 x \pm c_2 \\ b_3 y_3 = a_3 x \pm c_3 \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \end{gathered}$$

These types of equations are sometimes also known as conjunct pulveriser. He also solved the equations of Varga-Prakriti like $Nx^2 + 1 = y^2$. It is interesting to note here that Sripati (1039 A.D.) was able to recognise the infinite number of solutions of the equations of the form $Nx^2 + 1 = y^2$ and this very important discovery of Sripati was wrongly attributed by modern historians of mathematics to Fermat (1657), who also opined the same view in seventeenth century that the equation $Nx^2 + 1 = y^2$ has an unlimited number of solutions in integers, which was already made clear by Sripati even six centuries before than Fermat. Sripati also considered the rational solution of square nature for the equation of the form $Nx^2 + 1 = y^2$. He also gave the solutions for the equations of the form $axy = bx + cy + d$.

In Siddhantasekhara, Sripati explained the topics related to arithmetic, algebra, mensuration and few specific problems related to shadow reckoning. He described five methods of multiplication in his text as Kapata-sandhi, Tastha, Rupa-Vibhaga, Sthana-Vibhaga and Ista-Gunana. He gave the name Sadrsi-Karana (making similar) for equation and denoted the unknowns by colours like kalaka (black), nilaka (blue) etc. He also wrote two texts related to Jyotisla (astrology) titled as Jyotisharatnamala and Jatakapaddhati or Sripatipaddhati.

Sripati dealt with several problems related to geometry. He beautifully explained the method to calculate the circumradius of an isosceles trapezium.

चतुर्भुजेष्ट श्रुति पार्श्वबाहु वधस्य
चार्द्ध खलु लम्बभक्तम्
अतुल्यबाहोः प्रतिबाहुभाग
वर्गैक्यमूलस्य दलं हि यद्वा

(Sri Se p. 86)

According to this sloka, if we divide half the product of a diagonal and its adjacent side of any quadrilateral (isosceles trapezium) by the altitude, then we get the circumradius. But if a quadrilateral which is isosceles trapezium, has unequal sides then circumradius is half the square root of the sum of the squares of the opposite sides. Sripati also gave rules for the construction of rational right triangle and rational cyclic quadrilateral which are very similar to the rules given by Brahmgupta. Sripati explained the solution of rational right triangle in the sloka.

इष्टा भुजा तत्कृतिरिष्ट भक्त हीनाद्धिता
कोटिरसौ समेता
प्राग्भाजकेन श्रवस्सुधोभिर्जात्यो
यतः क्षेत्रविधौः निरुक्तः

(Sri Se p. 87)

According to this sloka, for given bhuja, if square of it is divided by any assumed number, again decreased by same number and then taking half of this value, it gives koti, whereas hypotenuse is found by addition of same quotient with the divisor (assumed value). *i.e.* for any side of length $m$, take assumed value $n$, sides of rational right angled triangle should be $m$, $\frac{1}{2}\left(\frac{m^2}{n}-n\right)$, $\frac{1}{2}\left(\frac{m^2}{n}+n\right)$. Sripati also gave the formula to find the common chord and its height in the case of intersecting circles whose diameters are $d_1$ and $d_2$ and heights of arcs $h_1$ and $h_2$ as

$$h_2 = \frac{\{d_1-(h_1+h_2)\}(h_1+h_2)}{d_1+d_2-2(h_1+h_2)}$$

He also gave the methods to find the value of segments of a circle and the value of $\pi$, similar to his predecessors.

Sripati also contributed much in the area of trigonometry. Sripati explained that the successive sines are found if differences of sines from a table are added in the direct order *i.e.* from top to, down. Similarly, the corresponding versed sines can be calculated if 'differences of sines are added in reverse order. He also expressed that the difference of sines as Jyakhanda, whereas the corresponding versed sines found in the reversed way from tabular

difference of sines are called as Vyasta-jya. Sripati explained few trigonometric results in following statements :

(*i*) the square of the radius is diminished by the square of the R sine, the square root of the remainder will be R cosine. Again the square root of the square of the radius minus the square of the R cosine will be the R sine. The radius minus the versed R sine of the complement of an arc is equal to the R sine of the arc and minus the versed R sine of the arc becomes the R sine of the other *i.e.* complement.

(*ii*) the chord of the sixth part of the circumference of a circle is equal to its semidiameter. The hypotenuse arising from the base and perpendicular (of a right angled triangle) each equal to the semi-diameter is the chord of the fourth part of the circumference. Half those chords will be the R sines of half of those arcs.

(*i* & *ii* translated by Datta & Singh)

Sripati constucted the table for R sines and versed R sines for every 3° 45′ from the circle of radius 3415′.

# 16. Bhaskara II (Bhaskaracharya)

Year of Birth : 1114 C.E.
Year of Death : 1185 C.E. (Probably)

Bhaskaracharya was a versatile mathematician of twelfth century A.D. The mathematics developed by Hindu mathematicians reached to its ac'me in the works of Bhaskaracharya and for long time his texts were followed throughout India as standard mathematical and astronomical texts, which were certainly in a position to sitmulate the curiosity of seekers in mathematics because of its attractive poetic presentation and elegant depiction of mathematical problems in the form of Sanskrit verses. He gave informations about his year of birth and time in this verse as

रसगुणपूर्णमही समशक नृप समये ऽभवन्ममोत्पत्तिः
रसगुण वर्षेण मया सिद्धान्तशिरोमणी रचित:

(Siddhanta Siromani)

According to it, he was born in 1036 Saka year and he composed Siddhanta-Siromani, when he was thirty years of age. With this, it can be computed easily that his year of birth in Christian Era should be 1114 C.E. *i.e.* (1036 + 78 = 1114) and in the year 1150 C.E. he wrote Siddhanta-Siromani. Regarding his native place and ancestral lineage he gave informations in this verse as

आसीत् सह्य कुलाचलाश्रितपुरे त्रैविद्यविद्व्ज्जने
नाना सज्जन धाम्नि विज्जडविडे शांडिल्यगोत्रो द्विज:

(Goladhyaya)

According to it, Bhaskaracharya was the Brahmin of Sandilya Gotra and his native place was at Vijjada-Vida, a place near to the Sahyadri mountains, which in the opinion of some scholars may be Bijapur of modern Karnataka, whereas few other scholars identified this place in Maharashtra. His father's name was Mahesvara, who himself was great scholar of mathematics and astronomy and

and he only taught the basics of these subjects to Bhaskara II. In the eulogy of his father he expressed that

आसीन्महेश्वर इति प्रथितः पृथिव्याम
आचार्यवयपदवीं विदुषां प्रपन्नः
लब्ध्वावबोधकलिका तत एव चक्रे
तज्जेन बीजगणितं लघु भास्करेण

(Bijaganita)

Bhaskaracharya also got education from his father in the area of Veda, Vyakarana, Jyotisa, Kavya, Literature etc. It is also believed that his grandfather and great grandfathers were also scholars of mathematics and astronomy and it was the natural gift only that for many generations, in his family excellent scholars were born and in one way or the other it became their vocation to study and develop the subjects related to mathematics and astronomy. His son was Laxmidhar and grandson was Changadeva. It becomes evidently clear through stone inscription found from Patan that Changadeva was a scholar and astronomer at the court of king Singam of Yadava dynasty, who ruled at Devagiri from 1210 A.D. to 1237 A.D. Changadeva also built a monastery to further develop and propagate the works of Bhaskaracharya. In one of the inscriptions Changadeva also mentioned that king Jaitrapala invited his father Laxmidhar, son of Bhaskaracharya, from Patan and from this information some scholars also conjectured that Bhaskaracharya's native place probably may be Patan (Khandesh). Bhaskaracharya later on migrated to Ujjain and became head of the astromical observatory there, to further develop and progress the mathematical tradition of Varahamihira and Brahmagupta.

The important works of Bhaskaracharya are Siddhanta Siromani, Karan Kutuhala and Vasanabhasya. Siddhanta Siromani consists of four parts namely Lilavati, Bijaganita, Goladhyaya and Grahaganita. Out of these four parts, Lilavati and Bijaganita exclusively cover the topics related to mathematics and Goladhyaya and Grahaganita deal with astronomy. Lilavati is divided into 13 chapters and covers the topics related to arithmatic, elementary algebra, mensuration and geometry. Bijaganita is an important treatise on algebra and covers the topics like positive and negative numbers, zero, unknown qunatities, surds, indeterminate equations of second, third and fourth degree, indeterminate quadratic equations etc. Goladhyaya contains thirteen chapters covering the top-

ics like nature of the sphere, planetry motion, spherical trigonometry, eccentric epicyclic model of the planets, ellipse calculations, the seasons etc. Grahaganita is divided into twelve chapters and deals with true and mean positions of the planets, diurnal rotation, Syzygies, Lunar and solar eclipses, latitudes of the planets, risings and settings etc.

In the year 1183 A.D., when he was of the age 69 years, he authored another astronomical text Karan Kutuhala, which gives details about lagna, kundli, phalit jyotisa and the methods to prepare alamanac. He also wrote commentary on his own text Siddhanta Siromani in prose from with title Vasanabhasya. On the text Shishyadhividdhidatantra of Lalla he wrote a commentary, named as Vivarna.

Lilavati of Bhaskaracharya is based on Brahmasphuta Siddhanta of Brahmagupta, Pattiganita of Sridhara and Maha-Siddhanta of Aryabhata I. He incorporated the important concepts enunciated by his predecessors, improved them where ever necessary and enriched his texts with his own pioneer results. Regarding the name Lilavati, it is said that Bhaskaracharya on the basis of his astrological calculations inferred that his daugther Lilavati has chances of becoming widow if marriage could not get performed at a particular time. He calculated that particular auspicious time for her marriage and to know that time correctly, made a water clock himself by placing a cup with a small hole at the bottom, on a pot filled with water. It was designed in such a way that after one hour the cup would sink. Lilavati because of her childlike curiosity stood near that instrument and started observing it, In the meantime, a pearl from her bridal dress fell into the cup and prevented the influx of water into the cup. The auspicious time passed without sinking of the cup, which forced Lilavati to remain whole life unmarried and left Bhaskaracharya in deep dejection. Finally, he consoled Lilavati with an immortal allurement that he would write a text on mathematics with title Lilavati, which will definitely make Lilavati immortal. Later on this text became very popular because of its coverage of relevant topics, poetic style, lucid presentation and highly imaginative coherence with natural settings. Lilavati was adopted as a standard textbook of mathematics throughout India for many centuries. It was translated into Persian by Faizi in 1587 C.E., on the directions of emperor Akbar. H.T. Colebrook translated Lilavati of Bhaskaracharya with title "Algebra with Arithmatic and Mensuration etc." and J. Tylor. translated it with title 'Lilavati'

itself in English. Several hundreds of commentaries were written by the scholars on Lilavati, Bijaganita and Siddhanta-Siromani. It is interesting to note that 143 copies of the commentaries written on Lilavati by different scholars are still available in India and abroad. The important commentaries are listed below :

| | | |
|---|---|---|
| 1. Paramesvara Ganesa | 1430 A.D. | Lilavativyakhya |
| 2. Daivajna | 1545 A.D. | Buddhivilasini |
| 3. Suryadasa | 1541 A.D. | Ganitamrtakupika |
| 4. Narayana | earlier than 1588 A.D. | Karmapradipaka |
| 5. Madhava | — | Karmapradipaka |
| 6. Gangadhara | 1432 A.D. | Ganitamrtasagari |
| 7. Ramakrsna | 1339 A.D. | Ganitamrtalahari |
| 8. Laxsmidasa | 1500 A.D. | Cintamani |
| 9. Munisvara | 1608 A.D. | Nisrstaduti |

Bhaskaracharya very frequently used the concept of zero and place value system. He expressed numbers in decimal scale in the multiple of ten as

एकदशशत् सहस्रायुतलक्षप्रयुत कोटयः क्रमशः।
अर्बुदम्बजं खर्वनिखर्व महापदम शंक वस्तस्यात्॥
जलधिश्यात्य मध्यं परार्धमिति दशगुणोत्तरा: सज्ञा
सखेयाया स्थानानां व्यावहारार्थ कृताः पूर्वैः॥

(Lilavati)

*i.e.* Eka (1), Dasa (10), Shata ($10^2$), Shasra ($10^3$), Ayuta ($10^4$), Laksha ($10^5$), Prayuta ($10^6$), Koti ($10^7$), Arbuda ($10^8$), Abja ($10^9$), Kharva ($10^{10}$), Nikharva ($10^{11}$), Mahapadma ($10^{12}$), Sanku ($10^{13}$), Jaladhi ($10^{14}$), Antya ($10^{15}$), Madya ($10^{16}$), Parardha ($10^{17}$). Bhaskaracharya gave rules for addition, subtraction and division. He explained five methods of multiplication namely Kapata-Sandhi, Tastha, Rupa-Vibaga, Sthana-Vibaga and Ista-Gunana. He used the notion of 'Ankanam Bamto Gati'. One of the methods of multiplication given by him is illustrated below

(*i*) Multiply 2146 with 13

26146 ; ($2 \times 13 = 26$)

27346 ; ($13 \times 1 = 13$, carry over 1 ; $26 + 1 = 27$)

27826 ; ($13 \times 4 = 52$ ; carry over 5 ; $273 + 5 = 278$)

27898 ; ($13 \times 6 = 78$ ; carry over 7 ; $2782 + 7 = 2789$)

In Lilavati, we find four different methods for getting the value of square. The first method is almost similar to the method followed in modern mathematics but other three methods are entirely different in terms of the algorithm involved. In sloka II of Lilavati he gave a rule to find the value of square, which is explained in following example :

**Example :** Find square of 3125

(*i*) $3^2 = 9$ ; $2 \times 3 = 6$ ; $6 \times 125 = 750$ ; $9 + (750) = 9750$

(*ii*) $1^2 = 1$ ; $2 \times 1 = 2$ ; $2 \times 25 = 50$ ; $1 + (50) = 150$

(*iii*) $2^2 = 4$ ; $2 \times 2 = 4$ ; $4 \times 5 = 20$ ; $4 + (2)0 = 60$

(*iv*) $5^2 = 25$

required value is

$$\begin{array}{r} 9750\phantom{000} \\ 150\phantom{00} \\ 60\phantom{0} \\ 25 \\ \hline 9765625 \end{array}$$

In Sloka 13 of Lilavati, he explained other method of finding the square that first divide given number into two parts, add sum of the squares of those parts with twice the product of parts. It can be put in the form of identity as $(a + b)^2 = a^2 + b^2 + 2ab$. The third method given in Lilavati is an interesting way to find the square, which is equivalent to the identity $x = (x + y)(x - y) + y^2$ ; where $y$ is an assumed number. It is illustrated below :

(*i*) Find square of 9897

Let assumed number be 103, so

$$(9897)^2 = (9897 + 103)(9897 - 103) + 103^2$$
$$= 97940000 + 103^2$$

Again for the square of 103, take assumed number 3, so that

$$103^2 = (103 + 3)(103 - 3) + 3^2 = 10600 + 9 = 10609$$

finally, $(9897)^2 = 97940000 + 10609 = 97950609$.

He also explained the rules to find square root and cube. To find cube root of a number he explained the rule of extraction in this verse,

आद्यं घनस्थानम थाघने द्वे पुनस्तथान्त्याद्धनतो विशोध्य
घंन पृथकस्थं पदमस्य कृत्या त्रिघन्या तदाऽऽद्यं विभजेत्फलं तुः
पकंत्यां न्यसेवत त्कृतिमन्त्यनिघ्नीं भिघ्नीं त्यजेत्तत्प्रथमा त्फलस्य
घंन तदाद्याद्धनमूलमेवं पंक्ति भर्वदेवमतः पुनश्चः॥

(Lilavati 14-15)

This method is explained in following example :

**Example :** Find cube root of 42875

(*i*) $\bar{4}\ \overset{1}{\underset{-}{2}}\ \bar{8}\ 7\ \overset{1}{\underset{-}{5}}$

First assign the cubic and noncubic places to the digits.

(*ii*) Given number has one triplet and other block of two digits from right to left, so its cube root has two digits.

(*iii*) Now take 42, which is closer to cube of

3 *i.e.*, 27, so

| | | |
|---|---|---|
| 42 | 8 | 75 |
| 27 | | |
| 15 | 8 | 75 |

(*iv*) By simple observation it can be ascertained that last digit of cube root will be five

(*v*) $3^2 = 9$ ; $9 \times 3 = 27$ ; $27 \times 5 = 135$

15875
135
2375

(*vi*) $5^2 = 25$ ; $25 \times 3 = 75$ ; $75 \times 3 = 225$

2375
225
125

(*vii*) $5^3 = 125$ ; 125
125
×

Required cube root is 35.

Bhaskaracharya while dealing with fractions considered only four types of Jatis. He explained the Rule of Three *i.e.* Trairasika and its inverse process and considered it indeed, the essence of arithmetic by expressing that "As Lord Narayana who relieves the sufferings of birth and death, who is the sole primary cause of the creation of the universe pervades this universe through His own manifestations as worlds, paradises, mountains, rivers, gods, men, demons etc. so does the Rule of three pervade the whole of the science of calculation ......" (Tr. by Datta and Singh) He also gave the rules of five, seven, nine or more terms. He solved many problems by using the rule of false position, which he named as

**istakarma** *i.e.*, the rule of supposition. We give some problems expressed by him related to it below.

अमल कमलराशेस्त्र्यशंपंच्चाशषष्ठैः
त्रिनयन हरि सूर्या येन तुर्य्येण चर्य्याः
गुरुपदमथ षड्भिः पूजितं शेष पद्मैः
सकल कमल संख्या क्षिप्रमाख्याहि तस्यः

(Lilavati)

*i.e.* (out of a heap of pure lotus flowers, a third, a fifth, a sixth were offered respectively to the Gods Shiva, Vishnu and Surya and a quarter was presented to Bhavani. The remaining six were given to the venerable preceptor. Tell quickly the number of lotuses)

(Tr. by Datta and Singh)

Bhaskaracharya only dealt with those problems related to quadratic equation in Lilavati which involve square root, the other cases were treated in Bijaganita. The following problems expressed by him are of the first category as

पार्थकर्णवधाय मार्गणगणं क्रुद्धो रणे सन्दधे
तस्यार्द्धेन निवार्य्य तच्छरगणं मूलैश्चतुर्भिर्हयान्
शल्यं षड्भिरथेषुभिस्त्रिभिरपिच्छत्रं ध्वजं कार्म्मुकं
चिच्छेदास्य शिरः शरेण कति ते यानार्जुन सन्दधैः

(Lilavati)

*i.e.* (Partha, irritated in fight shot a quiver of arrows to slay karna with half his arrows, he parried those of his antagonist with four times the square root of the quiver full, he killed his horses, hit Salya with six arrows, with three he demolished the umbrella, standard and bow and with one he cut off the head of his foe. How many were the arrows, which Arjuna let fly.).

(Tr. by Datta and Singh)

**Solution.** Let arrows used by Partha be $u$, then,

$$\frac{u}{2} + 4\sqrt{u} + 10 = u$$

or $$4\sqrt{u} = \frac{u-20}{2}$$

or $$u^2 - 104u + 400 = 0$$

or $$(u-100)(u-4) = 0$$

$u = 100$ or $4$ ; but answer should be 100.

The similar type of other problem he expressed as,

अलि कुल दल मूलं मालतीं यातमष्टौ
निखिल नवम भागाश्चालिनी भृगंमेकम्
निशि परिमललुब्धं पदममध्ये निरुद्धं
प्रति रणाति रणन्तं ब्रूहि कान्तेऽलिसंख्याम्

(Lilavati)

*i.e.* (the square root of half the number of a swarm of bees is gone to a shrub of jasmin and so are eight ninths of the whole swarm, a female is buzzing to one remaining male that is humming within a lotus, in which he is confined, having seen allured to it by its fragrance at night. Say, lovely woman, what is the number of bees)

(Tr. by Datta and Singh)

**Solution.** Let $u$ be total number of bees.

so, $$\sqrt{\frac{u}{2}} + \frac{8}{9}u + 2 = u$$

or $$\frac{u}{2} = \frac{(u-18)^2}{81}$$

or $2u^2 - 153u + 648 = 0$

or $(u-72)(2u-9) = 0$

$$u = 72 \quad \text{or} \quad \frac{9}{2}.$$

Total bees should be 72.

Bhaskaracharya analysed the concept of zero in his texts Lilavati and Bijaganita elaborately. He expressed about it in Lilavati that

योगे खं क्षेपसमं वर्गादौ खं भाजितौ राशिः
खहर स्यात्खगुणः खं गुणाश्चिन्त्यश्च शेषविधौ
शून्ये गुणके जाते खं हारश्चेत्पुन स्तदा राशिः
अविकृत एव सेयस्तथैव खेनोनितश्च युतः

(Lilavati)

According to it, in addition zero makes the sum equal to the number itself to which it is added, in involution and evolution the result is zero. A number divided by zero is kha-hera ; kha means sunya or zero *i.e.* number with zero as denominator. The product of a number and zero is zero but it must be put as a multiple of zero with a

distinction kha-guna. By analysing this explanation of zero some scholars opined that it is more likely to deduce the result that Bhaskaracharya was aware of the concept of infinitesimal qunatities. In their view, by making the assertion of kha-guna, he pointed making the assertion of Kha-guna, he pointed zero towards an infinitesimal quantity and applied this concept in astronomical calculations. While dealing with calculus he also used the quantities which ultimately approach to zero and derived the differential coefficients of certain functions by applying this notion. It is highly remarkable that Bhaskaracharya in twelfth century itself discovered the concept of infinitesimal quantities and applied them for various astronomical calculations which place him far ahead in terms of his mathematical acumen from other scholars. In fact, this notion of calculus, was conceived and developed in Western World only after six centuries of its introduction and application by Bhaskaracharya in India. To define the concept of infinity *i.e.* Kha-hara, which is synonymous with Kha-Cheda of Brahmagupta *i.e.* the quantity with value in denominator zero, Bhaskaracharya expressed that

अस्मिन विकार: खहरे न राशा
वपि प्रविष्टेष्वपि नि: सृतेषु
बहुष्वपि स्याल्लय सृष्टिकाले
ऽनन्तोऽच्युत भूतगणेषु यद्वत्

(Bijaganita 4)

*i.e.* (in this quantity, consisting of that which has Cipher for its divisor, there is no alteration, though many may be inserted or extracted ; as no change takes place in the infinite and immutable god, at the period of the destruction or creation of worlds, though numerous order of beings are absorbed or putforth)

(Tr. by Datta and Singh)

From this explanation, it is evidently clear that Bhaskaracharya was aware of the concept

$$\frac{p}{0} = \infty \quad \text{and} \quad \infty + q = \infty$$

He also made it clear that if any quantity is multiplied by zero and also divided by zero, in such a case its value remains unchanged *i.e.* $\frac{p \times 0}{0} = p$, which can also be expressed as $\lim_{\varepsilon \to 0} \frac{p \times \varepsilon}{\varepsilon} = p$. But one flaw is noticed in the explanation of Bhaskaracharya that he

considered zero itself as an infinitesimal quantity instead of ε or any other suitable notion meant to denote infinitesimal quantity as done in modern notations. In fact, scholars like Taylor and Bapu Deva Shastri made it clear that in $\left(\frac{p \times 0}{0} = p\right)$, Bhaskaracharya considered zero as a very small quantity equivalent to limiting value zero, which he applied further in astronomical calculations. He also gave three examples related to this concept as,

$$(i)\ \left[\frac{\left(x \times 0 + \frac{x \times 0}{2}\right)}{0}\right] = 63\ ;\ \text{giving}\quad x = 14$$

$$(ii)\ \frac{\left\{\left(\frac{x}{0} + x - 9\right)^2 + \left(\frac{x}{0} + x - 9\right)\right\}}{0} = 90\ ;$$

giving $x = 9$

$$(iii)\ \left\{\left(x + \frac{x}{2}\right) \times 0\right\}^2 + 2\left\{\left(x + \frac{x}{2}\right) \times 0\right\} \div 0 = 15\ ;$$

giving $x = 2$.

The values given by Bhaskaracharya in these examples are possible only when we consider $0 = \varepsilon$, a small quantity approaching towards zero.

Bhaskaracharya explained the topics related to permutations and combinations in detail. He also composed some interesting problems depending on it, one of such problems is given below

पाशाङ्कुशाहिडमरूक कपाल शूलैः
खट्वाङ्ग शक्ति शरचापयुतै भवन्ति:
अन्योन्य हस्तकलितैः कति मूर्तिभेदाः
शम्भोर्हरेरिव गदारि सरोजशङ्खैः

*i.e.* (how many variations of form of the god, Sambhu (Lord Siva) are possible by the arrangement in different ways of the ten items, held in his several hands, namely pasa (rope), ankusa (elephants hook), sarpa (serpent), of armaru (double drum), kapala (skull), sula (trident), khatvanga (bedstead), sakti (dagger), sara (arrow) and capa (bow) as also those of Hari (Lord Visnu) by the exchange

of gada (mace), cakra (discus), saroja (padma, lotus), and sanka (conch) ?)

(Tr. by S.Balachandra Rao)

**Solution.** As there are ten items in the hands of Sambhu, these ten items can be exchanged with each other in $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 3 \times 4 \times 2 \times 1$ ways, which gives the value 3628800, so a sculptor has to make such number of different idols of Sambhu, which presents different set of ten items in each idol. Similarly, Hari's four items can be arranged in $4 \times 3 \times 2 \times 1$ ways so in this case 24 different idols may be created.

Bhaskaracharya defined algebra as the science which deals with numbers expressed in the forms of symbols or unknowns. He also used varnas and colours to denote unknown quantities as Yavat-Tavat (so much as), kalaka (black), nilaka (blue), pita (yellow), lohita (red), haritaka (green), svetaka (white), citraka (variegated), kapilka (tawhy), pingalaka (reddish-brown), dhumraka (smoke-coloured), patalaka (pink), savalaka (spotted), syamalaka (blackish) etc. He explained the laws of signs as

(*i*) in the addition of two negative or two positive numbers the result is their sum, the sum of a positive and a negative number is their difference

(*ii*) a positive while being subtracted becomes negative and a negative becomes positive

(*iii*) the product of two positive or two negative is positive ; the product of positive and negative is negative.

(*iv*) the square of a positive and a negative number is positive ; the square root of a positive number is positive as well as negative. There is no square root of a negative number, because it is nonsquare. (*i* to *iv* Tr. by Datta and Singh)

He considered six fundamental operations in algebra as addition, subtraction, multiplication, division, squaring and getting square root. He classified equations in different categories as (*i*) linear equations with one unknown (*ii*) linear equations with more unknowns (*iii*) equations in one unknown in its second and higher degrees (*iv*) equations having two or more unknowns in their second and higher power (*v*) equations involving products of unknowns. Bhaskaracharya analysed the problems related to indeterminate nature more explicitly and solved several such type of equations first time. He gave solutions to the equations of the form $by - ax = \pm c$ ; $by = ax \pm 1$ ; $by + ax = \pm c$ and simultaneous indeterminate equations of the form $by_1 = a_1x \pm c_1$, $by_2 = a_2x \pm c_2$,

$by_3 = a_3 x \pm c_3$ ...... Bhaskaracharya solved the indeterminate quadratic equations of the form $Nx^2 + 1 = y^2$, which are named by Hindu mathematicians as Varga-Prakriti. It is appropriate to mention here that the equation $Nx^2 + 1 = y^2$ named as Pell's equation by mistake by Western historians of mathematics because it seems that they were unaware of the fact that such type of equations of indeterminate nature were solved by Hindu mathematicians centuries before it was rediscovered in the West. Brahmagupta first gave solutions of the equations of the form $Nx^2 \pm c = y^2$ and $Nx^2 \pm 1 = y^2$ in rational integers, which were further improved and refined by Sripati, Bhaskaracharya, Narayana etc. In the equation $Nx^2 \pm c = y^2$ ; N is termed as gunaka-prakrti ; $x$ : kanisthapada or adyamula ; $y$ : jyesthapada or anyamula ; C : ksepa or praksepka by Hindu mathematicians. Bhaskaracharya solved the equations of Varga-Prakriti by using Cakravala (cyclic) method. The Cakravala method of Bhaskara II is explained below :

Take any Varga–Prakriti $Nx^2 + P = y^2$

where $P = \pm 1, \pm 2$ or $\pm 4$

We find $r$ and $s$ such that

$Nr^2 + P = s^2$, for any suitable P.

We can also get

$N \cdot 1^2 + (q^2 - N) = q^2$ and by applying Samasa-Bhavana, it can be obtained

$$N\left(\frac{rq+s}{P}\right)^2 + \left(\frac{q^2-N}{P}\right) = \left(\frac{sq+Nr}{P}\right)^2$$

Here the value of $q$ can be choosen in such a way that $rq + s$ is divisible by P and $q^2 - N$ becomes numerically small. Then put,

$$\frac{rq+s}{P} = r_1 ; \frac{q^2-N}{P} = P_1 ; \frac{sq+Nr}{P} = s_1$$

which gives Bhaskaracharya's first theorem in the form $Nr_1^2 + P_1 = s_1^2$, where if $r_1$ is an integer then $s_1$ and $P_1$ are also integers. If the same process is repeated then another set of integers $r_2$, $s_2$ and $P_2$ can also be obtained putting the equation in the form $Nr_2^2 + P_2 = s_2^2$. Continuing this process upto finite number of repetitions, two integers $r_1$ and $r_2$ can be obtained in such a way that $Ni_1^2 + \lambda = i_2^2$, where $\lambda = \pm 1$ or $\pm 2$ or $\pm 4$, which is called Bhaskaracharya's second theorem on Varga-Prakriti. Bhaskaracharya gave particular solutions for five values of N *i.e.* 8, 11, 32, 61 and 67. It is interesting to note here that Fermat in 1657 A.D. sought the solution of

the equation $61x^2 + 1 = y^2$ from other fellow mathematicians, but none of them could give the solution of this equation, whereas same equation $61x^2 + 1 = y^2$ was solved by Bhaskaracharya five centuries before Fermat's open question, which is sufficient to understand the mathematical ingenuity of Bhaskaracharya. The solution of the equation $61x^2 + 1 = y^2$ given Bhaskaracharya is explained below :

$$61\,x^2 + 1 = y^2$$

Here, $61.1^2 + 3 = 8^2$, where $r = 1$, $P = 3$, $s = 8$

Now, take $q$ in such a way that

$$\frac{rq+s}{P} = \frac{q+8}{3} \text{ is an integer and}$$

$$\frac{q^2 - N}{P} = \frac{q^2 - 61}{3} \text{ is numerically small.}$$

Let $q = 7$, then $r_1 = \dfrac{rq+s}{P} = 5$ ; $P_1 = \dfrac{q^2 - N}{P} = (-4)$

and $s_1 = \dfrac{sq + Nr}{P} = 39$, therefore

$$61.5^2 - 4 = 39^2$$

which is of the form $Nx^2 + \lambda = y^2$, where $\lambda = -4$, so that

$$61\left(\frac{5}{2}\right)^2 - 1 = \left(\frac{39}{2}\right)^2$$

Now by Samasa Bhavana between

$\frac{5}{2}, \frac{39}{2}, -1$ and $\frac{5}{2}, \frac{39}{2}, -1$ we get $\frac{195}{2}, \frac{1523}{2}, 1$

Again by Samasa Bhavana between,

$\frac{195}{2}, \frac{1523}{2}, 1$ and $\frac{5}{2}, \frac{39}{2}, -1$, we obtain

$i_1 = 3805, i_2 = 29718, \lambda = -1$. By applying Samasa on this set by itself Bhaskaracharya got the solution, $x = 226153980$ and $y = 1766319049$, which are the least integral values of this equation. Considering the importance of Cakravala method of Bhaskaracharya the famous German mathematician Hankel said that "It (Bhaskaras Cakravala method) is beyond all praise : It is certainly the finest thing achieved in the theory of numbers before Lagrange".

Bhaskaracharya asserted that the solution of these Varga-Prakriti equations should be infinite which can be obtained by

repeated application of the principle of composition. He also gave the rational solutions of these type of equations. He presented the solutions of the equations of the form $Nx^2 \pm c = y^2$ ; $Mn^2x^2 \pm c = y^2$ ; $a^2x^2 \pm c = y^2$ ; $c - Nx^2 = y^2$ and $Nx^2 - k^2 = y^2$. The solution of the general indeterminate equation of the second degree first time found in the Bijaganita. Bhaskaracharya classified indeterminate equations into two forms as Sakrt Samikarana (single equations) and Asakrt Samikaran a (Multiple equations). He gave solutions of the quadratic indeterminate equation of the form $ax^2 + bx + c = y^2$ ; $ax^2 + bx + c = a'y^2 + b'y + c'$ ; $a x^2 + by^2 + c = z^2$ and $ax^2 + bxy + cy^2 = z^2$. It is worthmentioning here that solution of some of these types of equations were rediscovered in Europe by Euler (1733 A.D.) and Lagrange's (1767 A.D.). He also studied single interminate equations of higher degree like $ax^{2n+2} + bx^{2n} = y^2$ ; where $n$ is an integer ; cube pulveriser like $bx + c = y^3$ ; double equations of first or second or higher degrees and equations of the form $axy = bx + cy + d$. He also gave geometrical demonstration of the equation $axy = bx + cy + d$.



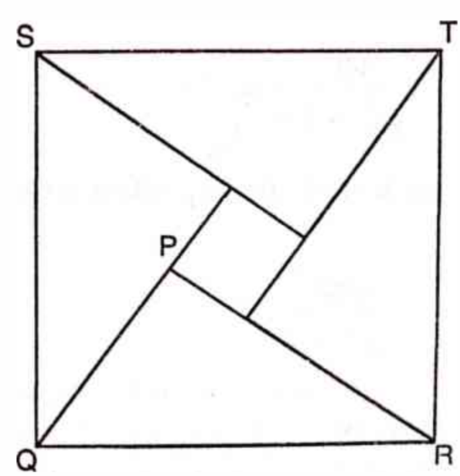
To construct a rational right triangle Bhaskaracharya gave the rule in this verse :

दो कोटयन्तरवर्गेण द्विघ्नो घात: समन्वित:
वर्गयोग सम: स स्याद् द्वयोर व्यक्त योर्यथा।।

(Bijaganita 129)

*i.e.* if square of the difference of bhuja and koti is combined with twice of their product then sum of their squares can be obtained.

To demonstrate it geometrically, take PQR, a right triangle, right angled at P, then four triangles similar to it are arranged together in such a way that hypotenuse of these triangles form the sides of square QRTS and form one small square at the centre with side equal to the difference of bhuja and Koti *i.e.* PR – PQ. Then area of all triangles is equal to

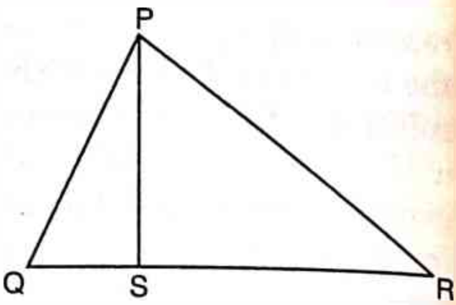



$$\left(4 \times \frac{1}{2} \times PQ \times PR = 2PQ \,.\, PR\right) \quad \text{and}$$

area of the bigger square = area of small square + area of all four triangles

*i.e.* $QR^2 = (PR - PQ)^2 + 2PQ \,.\, PR = PR^2 + PQ^2.$

For right angled triangle, he gave the following proof which was also rediscovered in West by Wallis in 1693 A.D.




Let PQR be right triangle right angled at P. Draw perpendicular PS on QR. Then the triangles QRP, QPS and PRS are similar. Therefore,

$$\frac{QR}{QP} = \frac{QP}{QS} \quad \text{or,} \quad QP^2 = QR \,.\, QS$$

Similarly, $\qquad PR^2 = QR \,.\, SR$

By adding these we get,

$$QP^2 + PR^2 = QR(QS + SR) = QR^2$$

Bhaskaracharya presented a new solution to find the sides of a right triangle whose one side containing the right angle is given, in this verse

इष्टो भुजोऽस्माद् द्विगुणेष्टनिघ्नादिष्टस्य
कृत्यैक वियुक्तयाप्तम्
कोटिः पृथक् सेष्टगुणा भुजोना कर्णो
भवेत्त्र्यस्रमिदं तु जात्यम्

(Lilavati)

Let base be given say $x$, if it is multiplied by twice of an assumed number say $p$, then Koti is $\frac{2xp}{p^2-1}$ and Karna (hypotenuse) is $\frac{2xp^2}{p^2-1} - x$. This result is further improved by taking given side as hypotenuse, then sides of right triangle should be $\frac{2xp}{p^2+1}$, $x - \frac{2xp^2}{p^2+1}$, $x$.

To construct a rational quadrilateral Bhaskaracharya expressed that two different rational right triangles are to be taken and from them two other triangles are to be constructed by multiplying the sides of these triangles by the hypotenuse of other triangle *e.g.* let triangles with given sides be (3, 4, 5) and (5, 12, 13), then other two triangles can be formed in this way : (3 × 13, 4 × 13, 5 × 13) or (39,

52, 65) and (5 × 5, 12 × 5, 13 × 5) or (25, 60, 65). The newly formed two triangles have the same hypotenuse and so these triangles can be juxtaposed with the common hypotenuse. This common hypotenuse will be one of the diagonals of quadrilateral and it will also be the diameter of circle inscribing this quadrilateral. Bhaskaracharya also gave several other rules related to various plane figures which look quite similar to those which were enunciated by his predecessors and simply can be considered reproduction of those rules.

To find the value of circumference of a circle and thereafter the gross value of $\pi$ Bhaskaracharya expressed that

व्यासे भनन्दाग्नि हते विभक्ते
खबाणसू परिधिः स सूक्ष्मः
द्वाविंशतिघ्ने विहृतेऽथ शैलेः
स्थूलोऽथ वा स्याद्वय वहारयोग्य।।

(Lilavati)

*i.e.* the sukshma value of circumference can be found when diameter is multiplied by 3927 and divided by 1250, but when diameter is multiplied by 22 and divided by 7, it gives the gross value of circumference.

Therefore $c = \dfrac{3920}{1250}\, d$ (precise value) and $c = \dfrac{22}{7}\, d$ (gross value). This result gives the suksma and gross value of $\pi$ as $\dfrac{3927}{1250}$ and $\dfrac{22}{7}$ respectively, where $\dfrac{3927}{1250} = 3.1416...$

Bhaskaracharya gave a rule to find the approximate value of chord in this verse as

चापोननिघ्नपरिधिः प्रथमाह्वयः स्यात्
पंचाहत् परिधिवर्ग चतुर्थ भागः
आद्योनितेन खलु तेन भजेच्चतुर्घ्नं
व्यासाहतं प्रथममाप्तमहि ज्यका स्यात्

(Lilavati)

*i.e.*, if $l$ is chord of the arc $a$, and $d$ diameter and $c$ circumference of the circle then

$$l = \frac{4d(c-a)a}{\frac{5c^2}{4} - (c-a)a}$$

He gave correct formula for the volume of a sphere in this verse

वृत्तक्षेत्रे परिधिगुणित व्यासपाद फलं य-
त्क्षुण्णं वेदैरुपरिपरित: कन्दुकस्येव जालम्
गोलस्यैव तदपि च फलं पृष्ठजं व्यासनिघ्नम्
षड्भिभर्क्तं भवति नियतं गोलगर्भे धनाख्याम्

(Lilavati)

According to it,

$$\text{area of a circle} = \text{circumference}\left(\frac{d}{4}\right)$$

$$= \frac{\pi \, . \, d^2}{4} = \pi \, . \, r^2$$

The surface area of a sphere

$$= 4 \text{ area of its great circle} = 4\pi r^2$$

$$\text{Volume of sphere} \quad = \text{surface area}\left(\frac{2r}{6}\right) = \frac{4}{3}\pi r^3$$

The method to find surface area of a sphere is explained by him in following verse :

गोलस्य परिधि: कल्प्यो वेदध्नज्यामितेर्मित:।
मुख बुध्नगरेखाभिर्य द्वदामलके स्थिता:।।
दृश्यन्ते वप्रकास्तद्वत् प्रागुक्त परिधिर्मितान्।
ऊर्ध्वाध: कृतरेखाभि: गोले वप्रान प्रकल्पयेत्।।
तर्भेकवप्रक क्षेत्रफलं खण्डै: प्रसाध्यते।
सर्वज्यैक्यं त्रिभज्यार्धहीनं त्रिज्यार्धभाजितम्।।
एवं वप्रफलं तत् स्याद गोल व्यास समं यत:
परिधि व्यास घातोऽतो गोलपृष्ठ फलं स्मृतम्

(Siddhanta Siromani, Goladhyaya 58-61)

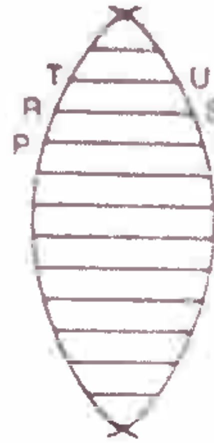
According to it,

the whole surface area of sphere should be equal to the product of the $d$ and $2\pi r$ *i.e.* $4\pi r^2$.

where diameter = area of one Vapraka

and area of one Vapraka

$$= \frac{\left(\Sigma r \sin n\alpha - \frac{r}{2}\right)}{\left(\frac{r}{2}\right)}$$




which is found by taking the sum of all trapezia like PQRS, RSTU and so on.

He enunciated beauifully the rule to find the volume of frustum of a pyramid in this verse :

मुखजतलजतद्युतिजक्षेत्रफलैक्यं हृतं षड्भिः

क्षेत्रफलं सममेवं वेधगुणं घनफलं स्पष्टम्

समखातफलत्र्यंशः सूचीखाते फलं भवति॥

(Lilavati)

From this verse, the volume of the frustum of a pyramid on a rectangular base with dimensons $p$ and $q$ of base and $p'$ and $q'$ of the top, and height $h$ should be equal to $h\left\{\frac{pq + p'q' + (p + p')(q + q')}{6}\right\}$ the volume of the frustrum of a cone should be equal to $\pi h \left\{\frac{d_1^2 + d_2^2 + (d_1 + d_2)^2}{6 \,.\, 4}\right\}$, where $d_1$ and $d_2$ are diameters of the base and top.

Volume of a pyramid with rectangular base should be $\frac{1}{3}(p \,.\, q \,.\, h)$.

Bhaskaracharya's contribution in the field of trignometry is also of high importance. He very clearly expressed the concept of jya equivalent to half chord and explained that the arrow between bow and bow string should be considered as versed sine. He made it clear that in a circle quadrants should be four and they should be odd and even successively. To get the values of R sine and R cosine he expressed that, take a point on the circumference of a circle, the perpendicular distance of that point from the east to west line is doh-jya *i.e.* R sine and its distance from north to south line is Koti-jya *i.e.* R cosine and corresponding arcs to these are called bhuja and Koti. If the point moves in anticlockwise direction the value of R sine increases and it becomes equal to the radius R at the end of

the quadrant, correspondingly the value of R cosine decreases and it becomes zero at the end of the quadrant. Bhaskaracharya formulated following trignometric identities and applied them to solve various problems

(*i*) $R^2 - (R \sin A)^2 = (R \cos A)^2$

(*ii*) $R \sin A = R \cos\left(\frac{\pi}{2} - A\right)$

(*iii*) $\frac{1}{2}\sqrt{\text{jya}^2\, A + \text{utkramajya}^2\, A} = \text{jya}\left(\frac{A}{2}\right)$

or $\frac{1}{2}\sqrt{\sin^2 A + \text{ver}\sin^2 A} = \sin\left(\frac{A}{2}\right)$

(*iv*) $\sqrt{\frac{R \times \text{utkramajya } A}{2}} = \text{jya}\left(\frac{A}{2}\right)$

or $\sqrt{\frac{\text{ver}\sin A}{2}} = \sin\left(\frac{A}{2}\right)$

(*v*) $\text{jya}\left(\frac{90 \pm A}{2}\right) = \sqrt{\frac{R^2 \pm R \times \text{jya } A}{2}}$

or $\sin\left(\frac{90^\circ \pm A}{2}\right) = \sqrt{\frac{1 \pm \sin A}{2}}$

(*vi*) $\frac{1}{2}\sqrt{(\text{jya } A - \text{jya.} B)^2 + (\text{kojya } A - \text{kojya } B)^2} = \text{jya}\left(\frac{A - B}{2}\right)$

or $\frac{1}{2}\sqrt{(\sin A - \sin B)^2 + (\cos A - \cos B)^2} = \sin\left(\frac{A - B}{2}\right)$

(*vii*) $\sqrt{\frac{(\text{jya } A - \text{kojya } A)^2}{2}} = \text{jya}\left[\frac{(90^\circ - A) - A}{2}\right]$

or $\sqrt{\frac{(\sin A - \cos A)^2}{2}} = \sin\left[\frac{(90^\circ - A) - A}{2}\right]$

(*viii*) $R \sin(90^\circ - 2A) = R - \frac{(R \sin A)^2}{(R/2)}$

(*ix*) $\text{jya}(A \pm B) = \frac{\text{jya } A \times \text{kojya } B}{R} \pm \frac{\text{jya } B \times \text{kojya } A}{R}$

or $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

These two formulas were named by Bhaskaracharya as Samasa-bhavana and antara-bhavana. He also gave the formulae for multiple angles which are equivalent to followings in modern trignometry.

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\sin 4\theta = 2 \sin 2\theta \cos 2 \theta$$

$$\cos 4\theta = \cos^4 \theta - 6 \sin^2 \theta \cos^2 \theta + \sin^4 \theta$$

$$\sin 3\theta = 3 \sin \theta \cos^2 \theta - \sin^3 \theta$$

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$$

$$\sin 5\theta = \sin 2\theta \cos 3\theta + \cos 2\theta \sin 3\theta$$

$$\cos 5\theta = \cos 2\theta \cos 3\theta - \sin 2\theta \sin 3\theta$$

$$\sin \theta = \sin 3\theta \cos 2\theta - \cos 3\theta \sin 2\theta$$

To find the value of R sine and R cosine for the angle 30° he stated that if a regular hexagon is inscribed in a circle then its side will be equal to the radius of the circle, so the value of R sin 30° should be half the radius. He gave following results as

$$\text{jya } 18° = \frac{1}{4}(\sqrt{5R^2} - R)$$

$$\text{jya } 36° = \sqrt{\frac{1}{8}(5R^2 - \sqrt{5R^4})}$$

Bhaskaracharya also gave a rule to compute the approximate values of $\sin\left(\frac{\pi}{n}\right)$ and stated that "multiply the diameter of a circle by 103923, 84853, 70534, 60000, 52055, 45922 and 41031 and divide the products by 12000, the quotient will be the sides of regular polygon inscribed in a circle from the triangle to the enneagon respectively" (Tr. by Datta & Singh)

According to it,

$$S_3 = D\left(\frac{103923}{120000}\right) = D \times (0.866025)$$

$$S_4 = D\left(\frac{84853}{120000}\right) = D \times (0.7071083)$$

$$S_5 = D\left(\frac{70534}{120000}\right) = D \times (0.587783)$$

$$S_6 = D\left(\frac{60000}{120000}\right) = D \times (0.5)$$

$$S_7 = D\left(\frac{52055}{120000}\right) = D \times (0.4337916)$$

$$S_8 = D\left(\frac{45922}{120000}\right) = D \times (0.382683)$$

$$S_9 = D\left(\frac{41031}{120000}\right) = D \times (0.341925)$$

From these, we can find following values:

$$\sin 60° = 0.866025$$

$$\sin 45° = 0.7071083$$

$$\sin 36° = 0.587783$$

$$\sin\left(\frac{\pi}{7}\right) = 0.4337916$$

$$\sin\left(\frac{\pi}{8}\right) = 0.382683$$

$$\sin\left(\frac{\pi}{9}\right) = 0.341925$$

It can be put in general form as

$$S_n = D \sin\left(\frac{\pi}{n}\right); \text{ where D is diameter.}$$

Bhaskaracharya also constructed the tables for R sines and versed R sines and their difference for the interval of 3° 45′ of a circle of radius 34 38′. He gave several methods to construct it. His first method is almost same as graphic method of Brahmagupta ; second method is also similar to Brahmgupta's theoritical method ; the third method is same as described by Aryabhatta II. The fourth method is devised by applying the formula

$$R \sin \frac{1}{2}(\alpha - \beta) = \frac{1}{2}\{(R \sin \alpha - R \sin \beta)^2 + (R \cos \alpha - R \cos \beta)^2\}^{1/2}$$

The fifth method is based on the formula

$$R \sin\left(\frac{\pi}{4} - \alpha\right) = \sqrt{\frac{1}{2}(R \cos \alpha - R \sin \alpha)^2}$$

To avoid the square root in computations as involved in all of the above mentioned five methods, he propounded a new method or sixth method by using the formula

$$R \cos 2\alpha = R - \frac{2(R \sin \alpha)^2}{R}$$

The seventh method of Bhaskaracharya to construct a table of twenty four R sines is expressed in this formula as

$$\text{jya}\,(n\alpha \pm \alpha) = \left(\text{jya}\,n\alpha - \frac{\text{jya}\,n\alpha}{467}\right) \pm \frac{100}{1529}\ \text{kojya}\,(n\alpha)$$

where $n = 1, 2, 3 \ldots\ldots 24, \alpha = 3° 45'$

and $\text{jya}\,\alpha = 225 - \frac{1}{7} = 224.856 \ldots$

In 1858 A.D. Pandit Bapu Deva Sastri first time expounded the fact that Bhaskaracharya used the notion of differential in Siddhantasiromani and it can be translated in modern notations as $\delta \sin \theta = \cos \theta\, \delta\theta$. This concept was developed by him to study the instantaneous motion of a planet and the position angle of the ecliptic in the chapter Spastadhikara of Siddhantasiromani. He described the concept of tatkalika-gati as instantaneous motion with reference to the planetary motion. This tat-kalika gati, he considered as suksma *i.e.* very small and the very small interval of time equivalent to a Ksena, which according to Hindu scholars is a quantity immeasurably small and can be compared with an infinitesimal interval of time. In fact, Bhaskaracharya, while taking the sine differences, introduced the notion of tat-kalika-bhogya khand *i.e.* instantaneous sine difference and expressed that

विम्बार्धस्य कोटिज्या गुणस्त्रि ज्याहरः
फलं दोर्ज्यायोस्तरम्॥

*i.e.* $\sin \omega' - \sin \omega = (\omega' - \omega) \cos \omega$

which can be put in modern notations as

$$\delta \sin (\omega) = \cos \omega\, \delta\omega$$

Bhaskaracharya also calculated the changes in the sine chord (bhuja jya) and cosine chord (kotijya) as

bhuja khanda or sine difference

$$= \frac{\text{cosine chord} \times \text{increase in the arc}}{\text{radius}}$$

Kotijyakhanda or cosine difference

$$= \frac{\text{sine chord} \times \text{increase in the arc}}{\text{radius}}$$

which can be transformed into modern notations as

$$d(\mathrm{R}\sin x) = \frac{\mathrm{R}\cos x \,.\, dx}{\mathrm{R}}$$

and $$d(\mathrm{R}\cos x) = \frac{\mathrm{R}\sin x \,.\, dx}{\mathrm{R}}$$

Bapu Deva Shastri presented the talkaliki motion of a planet of Bhaskaracharya in this way, let $x, x'$ be the mean longitudes of a planet on two successive days, $y, y'$ the mean anomalies, $u, u'$ the true longitudes and $a$ the sine of the greatest equation of the orbit, then

$$u' - u = x' - x \pm \frac{a \cos y}{\mathrm{R}} (y' - y)$$

or $$\delta u = \delta x \pm \frac{a \cos y}{\mathrm{R}} \delta y$$

The third important result related to differentials given by Bhaskaracharya is method of finding ayana-valana, where ayana-valana, is the sine of required angle *i.e.*

$$\mathrm{R}\frac{d\delta}{dl} = \frac{\mathrm{R}\cos l \,.\, \mathrm{R}\sin \omega}{\mathrm{R}\cos \omega}$$

where, $l$ and $\delta$ be celestial longitude and its corresponding declination of the ecliptic and $w$, the obliquity of the ecliptic.

Bhaskaracharya also presented two very important results which are now considered equivalent to mean value theorems as

(*i*) When a variable gets its maximum value, its differential becomes zero.

(*ii*) यत्रग्रहस्य परमं फलं

यत्रैवगति फलाभवेन भवितव्यम्

When a planet is either in apogee or in perigee, the equation of the centre vanishes, which in turn gives the result that, for some intermediate position the increment of the equation of the centre *i.e.* its differential also becomes zero. This second result is now considered as the clear exposition of the famous Rolle's theorem, the mean value theorem of differential calculus.

The earliest traces of integral calculus were found in the Goladhyaya of Bhaskaracharya, where he used the concept of integration to find the area of a circle, volume and surface area of the sphere. To find the area of a circle he described that if circle is

divided into large number of triangles with common vertex at the centre of circle and base on the circumference then by using the concept of summation of very large number of small units in the form of triangles, the value can be obtained. Similarly, he used the concept of summation to find the volume and surface area of sphere. It is again interesting to mention here that in these cases he was considering the summation of infinitesimal quantities and applying it in different setting to reach closer to the exact value.

# 17. Narayana Pandit

Narayana Pandit was probably born in 1340 A.D. His father was Narasimha. Though we do not have sufficient and authentic informations about the life history of Narayana Pandit but few inferences are made on the basis of available literature of that time and the kind of work done by him. Most of the scholars believe that he followed the tradition of Aryabhata school. He wrote the mathematical text "Ganita Kaumudi" which covers almost all aspects of mathematical knowledge of his time. He used different words for different numbers particularly successive multiples of ten like Eka (1), dasa (10), shata ($10^2$), sahasra ($10^3$), ayuta ($10^4$), laksha ($10^5$), prayuta ($10^6$), koti ($10^7$), arbuda ($10^8$), saroja ($10^9$), kharva ($10^{10}$), nikharva ($10^{11}$), mahapadma ($10^{12}$), shanku ($10^{13}$) etc.

In Ganita Kaumudi, he dealt with mathematical operations on numbers and gave an algorithm for multiplication of numbers and described a special method for squaring the number. He stated that square of the difference plus four times the product is the square of the sum. It can be expressed algebraically as $(a + b)^2 = (a - b)^2 + 4ab$. Narayana Pandit was the first Hindu mathematician to devise the rules for testing operations with the help of any desired number and clearly defined the rules to express the operations with zero. He also stated that the square of positive and negative number is positive, the square root of positive number will be positive and also negative. The negative number being nonsquare has no square root. For finding the square root of any algebraic expression he gave the rule that "first find the roots of square terms of given expression, then product of two and two of their roots multiplied by two should be subtracted from the remaining terms, result thus obtained is square root of given expression"

(Datta & Singh, page 28)

Narayana Pandit also solved many other algebraic equations of higher degree. For the equations like

$$x^2 + y^2 = p$$
$$x - y = q$$

He gave the solution as $x + y = \sqrt{2p - q^2}$

and $x = \frac{1}{2}\left(\sqrt{2p-q^2} + p\right)$ ; $y = \frac{1}{2}\left(\sqrt{2p-q^2} - p\right)$. Similarly, for the

equations $x^2 - y^2 = r$ ; $xy = s$ solution should be $x^2 = \frac{1}{2}(\sqrt{r^2 + 4s^2} + r)$

and $y^2 = \frac{1}{2}(\sqrt{r^2 + 4s^2} - r)$.

Narayana Pandit also solved indeterminate equations of the form $by - ax = \pm c$ and $Nx^2 \pm c = y^2$. He presented a beautiful method to find the approximate value of square root by using indeterminate equation of the type $Nx^2 + 1 = y^2$, where N is the number whose square root is to be calculated. To find the approximate value of squate root of N it is considered that if $x$ and $y$ are pair of the roots of equation $Nx^2 + 1 = y^2$, with the condition $x < y$, then square root of N should be equal to $\frac{y}{x}$. To demonstrate this notion, he took different pairs of roots to find the square root of 10, which has lot of mathematical importance in solving different problems. For the square root of 10, first he considered the value of $x = 6$ and $y = 19$, so $\sqrt{10} = \frac{19}{6}$ = 3.1666 666 666 666 666, which is correct upto two places of decimals. He then took the values of $x$ and $y$ as 228 and 721 respectively and

$$\sqrt{10} = \frac{721}{228} = 3.1622807017543859$$

which is correct upto four places of decimals. Finally he took the values of $x$ and $y$ as 8658 and 227379 respectively, so that

$$\sqrt{10} = \frac{227379}{8658} = 3.1622776622776622777$$

which is correct to eight places of decimals.

Narayana Pandit also expressed it in clear terms that for the equations of the form $Nx^2 - 1 = y^2$, solution is not possible, unless N is the sum of two squares.

*e.g.* $13x^2 - 1 = y^2$, here $13 = 2^2 + 3^2$, so its two rational solutions are $\left(\frac{1}{2}, \frac{3}{2}\right)$ and $\left(\frac{1}{3}, \frac{2}{3}\right)$. He also solved problems related to single indeterminate equations of higher degrees like.

(i) $x^3 + y^3 = x^2 + y^2$

(ii) $(x + y)^3 = xy$

He presented the solutions of these equations as

**Solution.** (i) $x = \dfrac{(p^2 + q^2)p}{p^3 + q^3}, y = \dfrac{(p^2 + q^2)q}{p^3 + q^3}$

(ii) $x = \dfrac{p^2 q}{(p+q)^2}$ ; $y = \dfrac{pq^2}{(p+q)^3}$

for any rational values $p$ and $q$.

In Ganita Kaumudi, Narayana Pandit also dealt with number sequences and solved many problems related to arithmetic progression. He also demonstrated the geometrical aspects of these arithmetic progressions by representing them by isosceles trapezia where the areas of these trapezia remain equal to the sum of the series. He also discussed in detail the rules for the formation of magic squares and other similar figures. The rules for the formation of even and odd perfect magic squares with magic triangles, rectangles and circles are also explained elaborately in this text. He also formulated a rule to estabilish the relation between magic squares and arithmetic series and presented the methods to find the horizontal and vertical difference by taking into consideration the first term of magic square and number of total terms in it. Narayana Pandit also gave the rules to find the areas, altitudes and diagonal of various geometrical figures like triangle, quadrilateral, trapezium, circle etc. He advanced further the analysis of cyclic quadrilateral by introducing the theorem of three diagonals as

सर्वचतुर्बाहूनां मुखस्य परिवर्तने यदा विहिते
कर्णस्तदा तृतीयः पर इति कर्णत्रयं भवति

(Ganita Kaumudi 48)

According to this sloka, the third diagonal in case of cyclic quadrilateral, named as para can be obtained, when the top side and flank side of any quadrilateral are interchanged. With the help of these three diagonals the area of cyclic quadrilateral can also be

calculated according to the rule given by Narayana Pandit in this verse

द्विगुण व्यास विभक्ते त्रिकर्णघातेऽथवा गणितम्

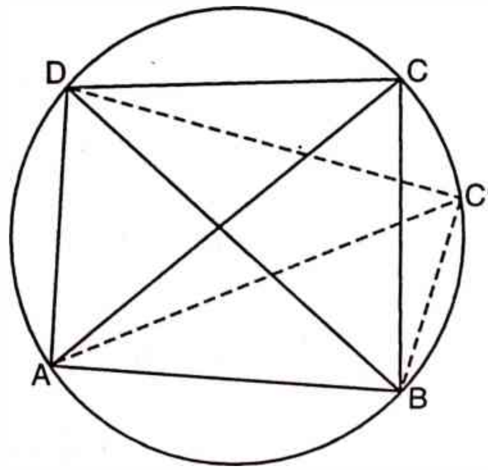
(Ganita Kaumudi 52)

*i.e.* if we divide the product of three diagonals by twice the circumdiameter we get the area of cyclic quadrilateral.

Area of quadrilateral ABCD

$$= \frac{AC.AC'.BD}{4r}$$

$$= \frac{AC.AC'.BD}{2d}$$



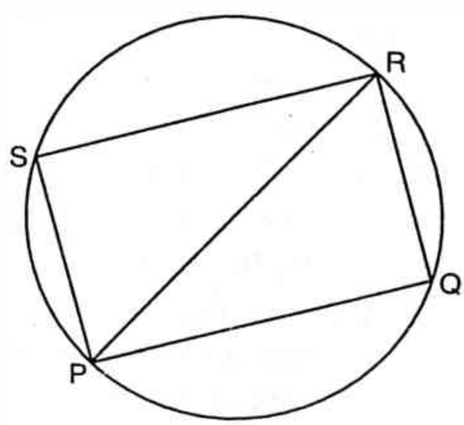

These three diagonals remain equal in a square and in that trapezium whose three sides are equal. Narayana Pandit also introduced a new type of trapezium named as Karanabhusana, in which the diagonals have the same length as the base. He gave various expression to find circumradius as

(*i*) Circumradius

$$= \frac{1}{2}\sqrt{\frac{(\text{Product of diagonals}) \times (\text{Product of flanks})}{\text{Product of altitudes}}}$$

(*ii*) Circumradius $= \dfrac{\text{the product of three diagonals}}{4 \times \text{area of quadrilateral}}$

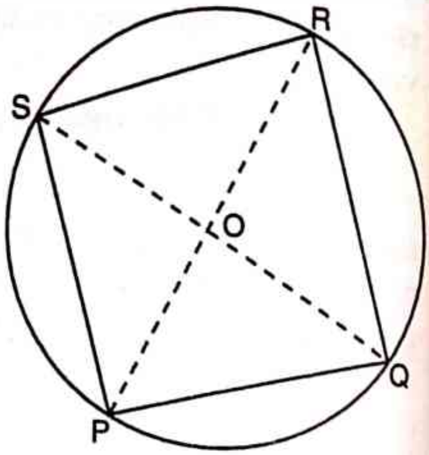
Narayana Pandit analysed the properties of quadrilaterals in detail and explained them in different verses in Ganita Kaumudi. These are given below in compact form




(*i*) area of quadrilateral PQRS

$$= \frac{PR\,(PS.SR + PQ.QR)}{4 \times \text{radius}}$$

(*ii*) If $a$, $b$, $c$, $d$ are the sides of given quadrilateral and $r$ is circumradius of first quadrilateral then the sides of second quadrilateral are $\sqrt{4r^2 - a^2}$, $\sqrt{4r^2 - b^2}$, $\sqrt{4r^2 - c^2}$ and $\sqrt{4r^2 - d^2}$.



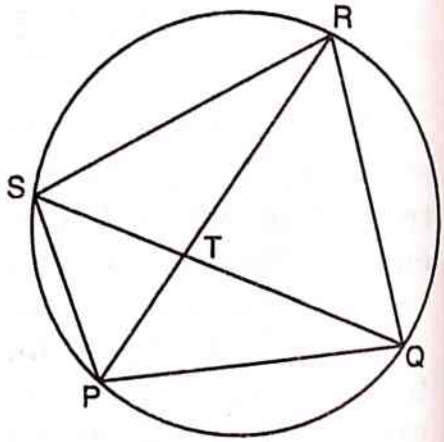

(*iii*)

$$PT = \frac{PS.PQ.PR}{PS.PQ + RQ.RS}$$

$$RT = \frac{RQ.RS.RP}{PS.PQ + RQ.RS}$$

$$QT = \frac{PQ.QR.QS}{PQ.QR + PS.SR}$$

$$ST = \frac{PS.SR.SQ}{PQ.QR + PS.SR}$$

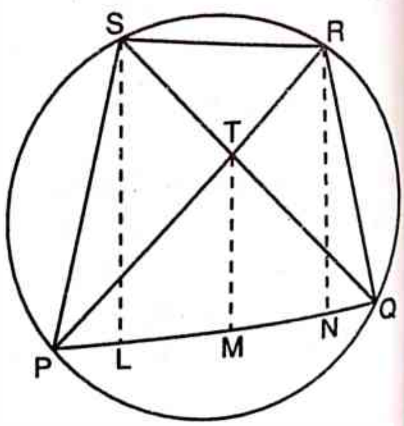



(*iv*)

$$SL = \frac{PT \times 2(\text{area of quadrilateral})}{PR \times PQ}$$

$$RN = \frac{QT \times 2(\text{area of quadrilateral})}{QS \times PQ}$$

$$TM = \frac{\sqrt{PT.QT.PS.QR}}{2(\text{circumradius})}$$




It is also believed that Narayana Pandit was also aware of the properties that

(*i*) the angle in a semicircle is a right angle

(*ii*) the angles in the same segments of a circle are equal.

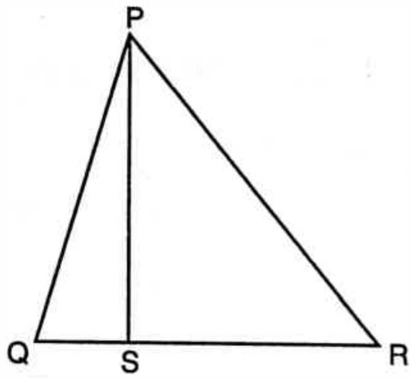
Narayna Pandit also gave a very interesting rule for the circumradius and area of triangle in Ganita Kaumudi as

अबधावधेन हीनो लम्बक वर्गोद्विलम्बक विभक्तः
तत्कृतिभूकृतियोगान्मूलदलं जायते हृदयम्॥

(Ganita Kaumudi 133)

According to this sloka circumradius should be equal to

$$\frac{1}{2}\sqrt{QR^2+\left(\frac{PS^2-QS.SR}{PS}\right)}$$




For the area of triangle he expressed that

चतुराहत हृदयहतं त्रिभुज भुजानां वधं गणितम्

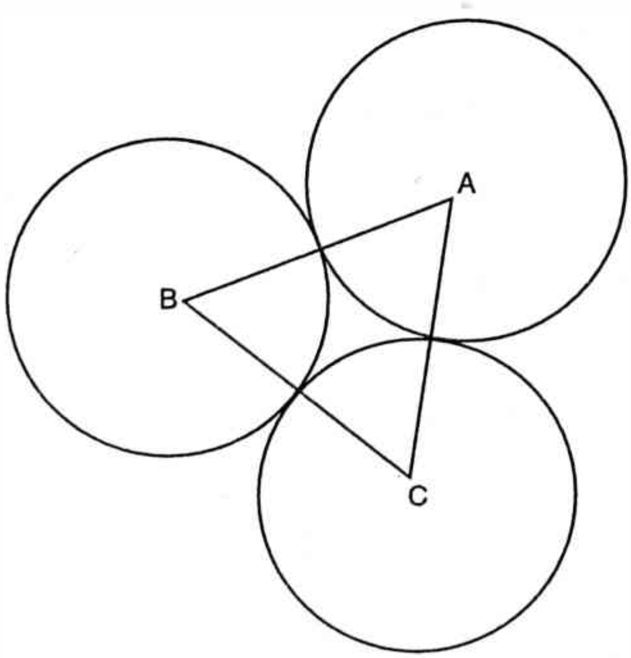
(Ganita Kaumudi 134)

According to this sloka, area of triangle should be equal to product of the sides of the triangle divided by four times the circumradius,

*i.e.* if $a$, $b$, $c$ are sides of the triangle and $r$ the circumradius then area of the triangle is $\frac{abc}{4r}$.

To find the area of space enclosed by three or more equal, mutually touching circles he gave the rule as

area of enclosed space is equal to the area of equilateral triangle ABC of side equal to the diameter of the circle minus half area of one circle.




Similarly, if four circles of equal radius mutually touch each other then area of enclosed space should be equal to area of square, with vertices at the centres of four circles and side equal to the

diameter of these circles minus the area of one circle *i.e.* $d^2 - \frac{\pi d^2}{4}$, where $d$ is diameter of circle.

Narayana Pandit also represented geometrically the arithmetic series even with fractional or negative periods. We describe below the rules given by him to demonstrate any arithmetic series in the form of trapezium and conversely, converting a trapezium into an arithmetic series. The rules given below are translated by T.S. Amma from the original verses given in Ganita Kaumudi

(*i*) the first term of the series diminished by half the common difference in the face, the product of the period and common difference increased by the face is the base, the period is the altitude and the area is the sum of the series. The fraction of the altitude multiplied by the common difference and combined with its own face is the base of any segment of the trapezium.

(*ii*) the base diminished by the face and divided by the altitude is the common difference ; the face combined with half the common difference is the first term, the altitude is the period and the area is the sum of arithmetic progression. If the face is negative, the first term and common difference can be obtained only if the altitudes at the middle is equal to the altitudes at the vertices, not otherwise in a quadrilateral of unequal sides.

Narayana Pandit also gave the rule for construction of integral triangles whose sides differ by one unit of length and which contain two right angled triangle of integral sides and common integral altitude. In case of inscribed polygons, he also stated that arc can be determined by dividing the circumference by the number of sides of that polygon.

# 18. Madhava of Sangamangrama

Madhava was born probably in 1340 A.D. in the South West Coast of Kerala, near Cochin. He was a versatile mathematician and astronomer from Kerala, who profoundly contributed in different branches of mathematics like, Mathematical Analysis, Calculus, Trigonometry, Geometry and Algebra. He was founder of Kerala School of Mathematics. Though, none of his mathematical texts are available now which are considered lost for some unknown reasons but his theorems, formulae, derivations, proofs, expositions and some profound idea of immense mathematical importance are found in the works of other scholars of later period particularly in the Yuktibhasa of Jyesthadeva, Tantra Sangraha of Nilakantha Somyaji and Kriyakramakari. Jyesthadeva gave the references of Madhava's theorems and other expositions in Yuktibhasa and explained them in vernacular language Malyalam. With these references, now it is understood that Madhava discovered and explained the concepts related to expansion of trignometric functions, approximations to rational value of $\pi$, methods of polynomial expansions, tests of convergence of infinite series, analysis of infinite continued fractions etc. He also constructed sine and cosine tables upto the accuracy of 12 and 9 places of decimals respectively. He generated the ideas related to differentials and integrals and solved many problems by using term by term integration. He also presented the iterative solution of transcendental equations. It is highly interesting that Madhava gave the expansions of trigonometric functions like $\sin x$, $\cos x$, $\tan x$, $\tan^{-1} x$ etc. around 1400 A.D., almost three hundred years before the period when the same concepts were conceived, introduced and rediscovered in Europe. These remarkable contributions of Madhava were also referred to in the works of his followers namely Mahajyanayana Prakara *i.e.* method of computing the great sines. In Yuktibhasa we find the references of sine and cosine series, which are attributed to Madhava. The sine series is explained in following verse as

निहत्य चाप वर्गेण चापं तत्तत् फलानि च
हरेत् समूलयुग्वर्गैस्त्रिज्यावर्गा हतैः क्रमात्
चापं फलानिं चा धोऽधौ न्यस्योपर्युपरि त्यजेत्
जीवाप्त्यै................................

(Quoted in Yuktibhasa p. 190)

On the basis of this sloka various terms can be obtained in the following manner, if $l$ is the arc and $r$ the radius then the values of these terms are

$$\frac{l \,.\, l^2}{(2^2+2)r^2} = \frac{l^3}{6r^2} = \frac{l^3}{3\,!r^2}$$

$$\frac{l^3 \,.\, l^2}{3\,!(4^2+4)r^4} = \frac{l^5}{120r^4} = \frac{l^5}{5\,!r^4}$$

$$\frac{l^5 \,.\, l^2}{5\,!(6^2+6)r^6} = \frac{l^7}{5\,!(7\,.\,6)r^6} = \frac{l^7}{7\,!r^6}$$

$$\frac{l^7 \,.\, l^2}{7\,!(8^2+8)r^8} = \frac{l^9}{7\,!(9\,.\,8)r^8} = \frac{l^9}{9\,!r^8}$$

Hence sine chord is equal to

$$l - \frac{l^3}{3\,!r^2} + \frac{l^5}{5\,!r^4} - \frac{l^7}{7\,!r^6} + \frac{l^9}{9\,!r^8} \;.....$$

Similarly, the versed sine is explaind in following verse

निहत्य चापवर्गेण रुपं तत्तत् फलानि च
हरेत् विमूलयुग्वर्गैस्त्रिज्यावर्गा हतैः क्रमात्
किन्तु व्यास दलेनैव द्विघ्नेनाद्यं विभज्यताम्
फलान्य धोऽधक्रमशोः न्यस्योपर्युपरि त्यजेत्
शराप्त्यै ..............................

(Yuktibhasa)

According to this verse, the expansion of versed sine should be equal to

$$\frac{l^2}{2\,.\,r} - \frac{l^4}{2\,.\,r^3\,.\,12} + \frac{l^6}{2\,.\,12\,.\,r^5\,.\,30} \;.....$$

or $$\frac{l^2}{2!r} - \frac{l^4}{4!r^3} + \frac{l^6}{6!r^5} \ldots\ldots$$

where $l$ is the arc and $r$ the radius.

In these above two expansions if we put $l = r\theta$, then we get

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} \ldots\ldots$$

and for cosine-chord, the value should be

$$r - \frac{l^2}{2!r} + \frac{l^4}{4!r^3} - \frac{l^6}{6!r^5} + \ldots\ldots$$

and putting $l = r\theta$, we get

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\ldots$$

Considering that the sine and cosine chords are $r \sin \theta$ and $r \cos \theta$ respectively.

It is pertinent to mention here that the power series expansions of $\sin x$ and $\cos x$ can also be obtained directly by following the Karanapaddhati system, in which Madhavas's verses can also get expressed directly and series are found straightway. In Kriyakarmakari also we find a verse which expresses an infinite series in the powers of sine chord and cosine chord for any given arc. The discovery of this very important result is also attributed to Madhava in this text. It is quiet interesting to note that this result was rediscovered in Europe after three centuries of Madhava's discovery, by Gregory (1671 A.D.) and Leibnitz (1673 A.D.), who presented this result in the form

$$\tan^{-1}\theta = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} \ldots\ldots \text{ to infinity}$$

Madhavas's verse regarding this series, quoted in Kriyakramakri is :

इष्ट ज्यात्रि ज्ययोर्घातात् कोटयाप्तं प्रथम फलं
ज्यावर्गं गुणकं कृत्वा कोटिवर्गं च हारकम्
प्रथमादिफलेभ्योऽथ नेया फलततिर्मुहुः
एकत्या द्योज संख्याभिर्भक्तोष्वेतेष्वनु क्रमात्

ओजानां संयुतेस्त्यक्त्वा युग्मयोगं धनुर्भवेत्
दोः कोटयोरल्पमेवेष्टं कल्पनीयमिह स्मृतम्॥

(Kriyakramakari p. 692-693)

According to this sloka, if S and C are the sine and cosine chords respectively, then the terms in the expansion of the arc should be like

$$\frac{Sr}{C};\ \frac{Sr}{3C}\cdot\frac{S^2}{C^2};\ \frac{Sr}{5C}\cdot\frac{S^4}{C^4};\ \frac{Sr}{7C}\cdot\frac{S^6}{C^6};\ \frac{Sr}{9C}\cdot\frac{S^8}{C^8};\ \frac{Sr}{11C}\cdot\frac{S^{10}}{C^{10}};\ \ldots..$$

Further these terms to be combined in such a way that sum of the terms in even places is subtracted from the sum in the odd places to get the arc. *i.e.*

$$\text{arc} = \left(\frac{Sr}{C}+\frac{1}{5}\frac{S^5}{C^5}r+\frac{1}{9}\frac{S^9}{C^9}r+\ldots\right)$$

$$-\left(\frac{1}{3}\frac{S^3}{C^3}r+\frac{1}{7}\frac{S^7}{C^7}r+\frac{1}{11}\frac{S^{11}}{C^{11}}r+\ldots\right)$$

or

$$\text{arc} = \frac{S}{C}r-\frac{1}{3}\left(\frac{S^3}{C^3}r\right)+\frac{1}{5}\left(\frac{S^5}{C^5}\right)r-\frac{1}{7}\left(\frac{S^7}{C^7}\right)r$$

$$+\frac{1}{9}\left(\frac{S^9}{C^9}\right)r-\frac{1}{11}\left(\frac{S^{11}}{C^{11}}\right)r+\ldots.$$

Since, $\frac{S}{C}$ = tangent of an arc, and then in a circle of unit radius,

$$\text{arc} = t-\frac{1}{3}t^3+\frac{1}{5}t^5-\frac{1}{7}t^7+\ldots..$$

where $t = \tan\theta$.

Madhava gave very interesting rational approximations for $\pi$ in the form of power series. This result is also given in Yuktibhasa as

$$C = 4d-\frac{4d}{3}+\frac{4d}{5}-\frac{4d}{7}+\ldots \text{ to infinity}$$

where C is circumference and $d$ diameter of a circle and $\pi = \frac{C}{d}$

$$\Rightarrow \qquad \pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \ldots \text{ to infinity}$$

$$\Rightarrow \qquad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \text{ to infinity}$$

this value of $\frac{\pi}{4}$ can also be obtained from the series

$$\text{arc} = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots \text{ to infinity}$$

where $t = \tan\theta$

or
$$\tan^{-1} t = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots\ldots \text{ to inifinity}$$

put $t = 1$, we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots \text{ to infinity}$$

One another approximation of $\pi$ given by Madhava is in the form of word numerals, which is quoted in Kriyakramakari as

विबुधनेत्रगजाहिहुताशनत्रिगुण वेदभवारण बाहवः

नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः

(Kriyakramakari Page 668)

According to this verse, if diameter of a circle is 900 000 000 000, then measure of its circumference should be 2, 827, 433, 388, 233. With this explanation it is clear that

$$\pi = \frac{\text{Circumference}}{\text{diameter}} = \frac{2{,}827{,}433{,}388{,}233}{9{,}00{,}000{,}000{,}000}$$

$\therefore \quad \pi = 3.14159265359$, which is correct upto ten places of decimals. Madhava also expressed another series for $\pi$ as,

$$\pi = \sqrt{12}\left[1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \ldots \text{ to infinity}\right]$$

From this series he computed the value of $\pi$ correct upto eleven places of decimals. But with these quite closer approximations of $\pi$ also, Madhava was not satisfied and he wanted to devise a formula which can approach very nearer to the exact value of $\pi$, for which he introduced one additional term $R_n$ in the expansion of $\frac{\pi}{4}$ as,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} \ldots + \frac{1}{(2n-1)} \pm R_n$$

where the value of this additional term $R_n$ he fixed either of these three *i.e.*

$$\frac{1}{4n} \quad \text{or} \quad \frac{n}{4n^2+1} \quad \text{or} \quad \frac{n^2+1}{4n^3+5n}$$

With these results it becomes clear that Madhava was a brilliant mathematician, who introduced and discovered several concepts of modern mathematics centuries before they became the part of treasure of modern mathematical world, with the false notion of their Western origin. In fact Madhava's exemplary work needs to be reexplored and due credit should be given to him for such pioneering work in different branches of mathematics. To substantiate this point here it would be appropriate to quote Joseph, about the achievements of Madhava as "we may consider Madhava to have been the founder of mathematical analysis. Some of his discoveries in this field show him to have possessed extraordinary intuition ; making him almost the equal of the more recent intuitive genius Srinivasa Ramanujan, who spent his childhood and youth at Kumbakonam not far from Madhava's birth place". It is really unfortunate that even with such an extraordinary work to his credit Madhava remained in oblivion for long in the eyes of modern mathematicians throughout the world. Though it is also true that his major contributions in the areas of mathematics could not survive and whatever came to the fore, it was only due to the references, quotations, proofs, derivations, expositions etc given by his successors in their own mathematical text, with a great sense of honesty by attributing such references to the original contributor *i.e.* Madhava. Rajagopal and Rangachari, considering Madhava's remarkable achievements in the field of mathematics expressed their views in these words, "Madhava took the decisive steps onwards for the finite procedures of ancient mathematics to treat their limit passage to infinity, which is the kernal of modern classical analysis".

# 19. Parameswara

Parameswara was born probably in 1370 A.D. at a village Alattur of Kerala state. He was the follower of Asvalayana Sutra of Rgvedin and belonged to the Bhrgu Gotra. His house Vatasseri was situated towards the North bank of river Nila, nearer to the confluence of Nile with the Arabian sea. From the works of Parameswara it is believed that Rudra was his teacher and he also got teachings from other great scholars of mathematics and astronomy particularly Madhava and Narayana. With the influences of these great scholars Parameswara later on became an important mathematician to extend the legacy of Kerala School of Mathematics. He enriched the treasure of Kerala school of mathematics and wrote more than thirty books related to astronomy and mathematics. Out of these his important contributions in the area of astronomy are the Drgganitam, three texts on spherics named as Goldipika volume I to III, three texts related to eclipses namely Grahanastaka, Grahanamandana and Grahana-Nyaya-Dipika, one text related to the shadow of moon named as Candra-Cchaya-Ganitam and a text related to the calculations on the mnemonic tables named as Vakyakarana. Parameswara also wrote number of commentaries on the works of his predecessors in the field of mathematics. Out of which the most celebrated commentaries written by him are related to the texts Aryabhatiyam of Aryabhata ; Mahabhaskariyam and Laghubhaskariyam of Bhaskara I ; Lilavati of Bhaskara II, Laghumanasam of Manjula, Surya Siddhanta, Vyatipatastaka, Goladipika I and Siddhantadipika. These commentaries are considered very useful even today by scholars to understand the various concepts of mathematical and astronomical importance. The commentary of Parameswara on the Manjulas astronomical work "Laghumanasam" contains topics like the mean motions of the heavenly bodies, the systems of coordinates, direction, place and time, eclipses of the

sun and the moon, the operation for apparent longitude etc. In the commentary on Arybhatiyam of Aryabhata, Parameswara gave several examples to determine the height of different vertical objects from the lengths of their shadows. One of the remarkable contributions of the Parameswara was related to the version of mean value theorem which he explained in his commentary Lilavati Bhasya on the text Lilavati of Bhaskara II. It seems that this result he could obtain because of the influence of Madhava's teachings on him. Govindasvami wrote one commentary the Bhasya, on the Bhaskara I's work Mahabhaskariya. Parameswara further wrote a commentary Siddantadipika on the Bhasya of Govindasvami, in which he gave interactive techniques to calculate the sine of a given angle with mean value formula to find inverse interpolation of the sine. In this text he also presented some of his observations on eclipses made at Navaksetra in 1422 A.D. and at Gokarana in 1425 A.D. and 1430 A.D. Parameswara made number of observations on the eclipses from 1393 A.D. to 1445 A.D., and on the basis of these observations he made some revisions in the planetary parameters. Parameswara followed a rigorous exercise for repeated observations, experimentation recording and checking of the observations, comparing these observed values with actual values and computing the margin of difference, analysing these observations and differences with the predetermined postulates and then making necessary corrections to reduce the margin of difference between actual and experimented values. In Drgganitam, he clearly stated that the actual positions of the planets seen by eye differ from the computed value of the positions of these planets according to Parahita system. Taking note of such flaws on the prevailed Parahita system of computations of that time, he suggested necessary corrections in due course of time by saying that.

कालान्तरे तु संस्कारश्चिन्त्यतां गणकोत्तमैः

Paramesvara while doing extensive work in Grahanmandana observed that the values obtained through calculations and observed positions generally differ. Paramesvara when got fully convinced that the existing Parhita system propounded by Haridatta centuries before, cannot give correct values and there is always a scope of getting differences in the computed values and actual values, then he introduced his own system of computation in his text Drgganitam, named as Drk system. In this Drk system he improved the multipliers and divisors and made minute corrections for the position of planets. He also made many corrections in respect of

the mean longitudes of the sum, the moon, mandooca and the node. The values of mandajya (sines of arc) and sighra-jya (conjunction) were also revised and then calculated for the interval of 6°. The observations and experimentations of Paramesvara in the field of astronomy are remarkable. It is interesting to note here that researchers of present age also have accepted the fact that Parameswara was the first astronomer who gave the idea about the heliocentric model of planets., which was introduced in the West by Copernicus even many years after to the period of Paramesvara.

Paramesvara also explained several theorems, rules and propositions enunciated by his predecessors in his commentaries. He analysed the properties related to quadrilateral in general, and cyclic quadrilateral in particular with a deep sense of insight. Regarding quadrilaterals he expressed that

यास्मिन् चतुर्भुजे एवं प्रकल्पितं कर्णद्वयं भवति
तत्र सर्वदोर्युति दलमित्यादिना आनीतं क्षेत्रफलं
च स्फुटं भवतीति। एतत्क्षेत्रफलान्न्यूनमेव
कर्णान्तरकृत क्षेत्रफलानि भवन्ति।

This makes it clear that the exact area of any quadrilateral can be calculated by applying the formula Sarvadoryutidala etc. *i.e.*

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Paramesvara further made it clear in other verse that constructing two rational right angled triangles with the four sides of any quadrilateral, the three diagonals can also be calculated with their help; for which the longest and least sides of quadrilateral should be divided by any arbitrary number, the value of these quotients will give the base and perpendicular side respectively. With the help of these two *i.e.* base and perpendicular, hypotenuse can be calculated, which produce one Jatya *i.e.* rational right angled triangle. Now by the hypotenuse of first Jatya, other two sides of quadrilateral should be divided, which will give the base and perpendicular of other jatya. Further, from these two Jatyas the diagonals of quadrilateral are calculated and from all four vertices of this quadrilateral circle also passes.

Parameswara gave one interesting formula to find the circumradius of a cyclic quadrilateral in terms of its sides in his commentary on the Lilavati. He was the first mathematician who established this relation in between the sides and circumradius,

**and this result was rediscovered by Lhuilier in 1782 A.D. more than two centuries after in Europe. The verse related to this formula is**

द्रोष्णां द्वयोर्द्वयोर्घातयुतीनां तिसृणां बधात्
एकै कोनेतरभ्यैक्यचतुष्क- वध - भाजिते
लब्धमूलेन यद्वतं विष्कम्भार्द्धेन निर्मितम्
सर्वं चतुर्भुज क्षेत्रं तास्मिन्ने वातिष्टते।।

According to this verse, if $a$, $b$, $c$ and $d$ are the sides of any quadrilateral then the product of two sides taken at a time and their sum, resulting in total three sums of this form, multiplied together gives $(ab + cd)(ac + bd)(ad + bc)$. Further, the four sums of sides, in which three are added at a time and diminished by fourth and then multiplied together gives the value $(a + b + c - d)(b + c + d - a)(c + d + a - b)(d + a + b - c)$. Again if the first product is divided by second product then the square root of this quantity will be radius and if a circle is drawn with this radius then whole quadrilateral will be situated within the circle having its vertices on the circumference of the circle

$$\text{Circumradius } (r) = \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(a+b+c-d)(b+c+d-a)(c+d+a-b)(d+a+b-c)}}$$

precisely if $u = (ab + cd)(ac + bd)(ad + bc)$

$v = (a + b + c - d)(b + c + d - a)(c + d + a - b)(d + a + b - c)$ then $r^2 = u/v$.

# 20. Nilkantha Somayaji (1444–1544 A.D.)

Nilkantha Somayaji was a great astronomer and mathematician. He is considered now an ardent exponent of Aryabhata school. Nilakantha Somayaji was Nambudiri Brahmin of Gargya gotra and his family people and ancestors were the follower of Asvalayana Sutra of Rgveda and worshipped the deity Soma, who is considered as master of plants and the healer of diseases. He was adorned by the title Somayaji because of being the devotee of Soma and this title sometimes also pronounced as Somabut or Somabutvan or Comatiri in Malyalam. He was born on June 17, 1444 in the village Trikkansliyur near Tirur in Kerala. He was educated in Astronomy, Mathematics and Vedanta by most respected teachers Ravi and Damodara. Guru Ravi taught him Vedanta and some basics of Astronomy whereas, Guru Damodara taught him Mathematics and Astronomy. It is interesting to mention here that Guru Damodara was the son of another great astronomer Parameswara, who propounded the Drk system of astronomy and was also the follower of the Asvalayana Sutra of Rgveda. Damodara trained Nilakanth Somayaji in the teachings of his father Parameswara. It prompted Nilkantha to follow the path of Parameswara and move further in that direction. He considered the basics of Parameswara vital in the field of Mathematics and Astronomy. Nilakantha also referred Parameswara at several places and called him very reverentially, Param Guru. Nilakantha Somayaji authored number of texts related to Astronomy and Mathematics namely

(*i*) Tantra Sangraha, containing 432 verses and centred towards Astronomy and Mathematics

(*ii*) Bhasya on Aryabhatiyam, a detailed commentary on Aryabhata's work

(*iii*) Golabara, concentrating on Spherical Astronomy

(*iv*) Siddhanta-Darpana

(*v*) Chandra-Cchaya-Ganitam, dealing with calculations related to shadow of moon

(*vi*) Sundararaja Prasnotara, giving details of the answers to the questions of Sundararaja related to astronomy.

Out of these works Tantra Sangraha is major and important contribution of Nilakantha Somayaji in the field of Astronomy and Mathematics. This text now is an authentic source to get acquainted with the discoveries of Madhava another illustratious mathematician of Kerala, whose original texts could not be traced till this time. Nilakantha referred Madhava's rules, methods, theorems and verses in his texts and wherever necessary, extended the domain of the results discovered by Madhava and in many cases he also improved the results of Madhava, so that they could be applied in different situations appropriately. The results of Madhava and Nilakantha Somayaji were further referred and proved by another important mathematician of later period Jyesthadeva in his text Yuktibhasa.

Tantra Sangraha covers various aspects of Hindu astronomy and primarily based on the epicyclic and eccentric models of planetary motion. It deals with the motions and longitudes of the plants, sun's position on the celestial sphere and its relationships with the three systems of coordinates *viz.* ecliptic, equatorial and horizontal. He also covered other aspects of the eclipses of the sum and the moon, deviation of the longitudes of the sum and the moon rising and meetings of the moon and planets and finally the graphical representation of size and part of the moon illuminated by the sum. Nilakantha's another work is Aryabhatiyabhasya, which is a commentary on the Aryabhatiya of Aryabhata I. In this Bhasya, he referred two eclipses, which he observed on 6th March 1467 and 28th July 1501 at Anantaksetra. In Golasara Nilakantha expressed the rules to use mathematical concept to calculate astronomical dates in fifty six Sanskrit verses; whereas Siddhanta Darpana is the text of Nilakantha, in which he described in detail the planetary system. The Chandra-Cchaya-Ganita is another text of Nilakantha, where he explained the methods of computations to find the moon's zenith distance.

Current researches are also revealing the important contributions done by the scholars of Kerala School of Mathematics in the fields of planetary theory. Nilkantha Somayaji made significant revision of the older Indian planetary model for the inferior planets. He formulated correctly the equation of centre for Mercury and Venus and described a heliocentric model of planetary

motion. With this achievement Nilakantha (1444-1544 A.D.) took a great stride in the area of astronomy even prior to Copernicus (1473-1543 A.D.) who is known to be the first western scholar to introduce the theory of heliocentricity. But it is interesting to note here that Nilakantha got the clue about this revolutionary concept of planetary motion from his predecessor, Parameswara (1360-1545 A.D.).

In Tantra Sangraha, Nilakantha gave the result of Madhava related to the following trigonometrical identity

$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$

But he also presented an alternative proof for this by expressing :

[illegible]

(Tantra Sangraha II)

According to this sloka, sine chord of the sum or difference arc can be found by combining the roots of the differences of the squares of the sine and its altitude. He also referred many other verses of Madhava related to trigonometric functions and their expansions in the form of power series. Nilakantha used the concept of limit to find an approximate expression for an arc of the circumference of a circle, from which finally he derived the expansion of arc (tan x) i.e.,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

From this series, the value of π/4 can also be obtained by putting x equal to 1 i.e.,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Nilakantha Somayaji also found many interesting series expansions of π, like :

$$\frac{\pi}{8} = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \ldots$$

$$\frac{\pi - 3}{4} = \frac{1}{2.3.4} - \frac{1}{4.5.6} + \frac{1}{6.7.8} - \frac{1}{8.9.10} + \ldots$$

$$\frac{\pi}{16} = \frac{1}{1^5 + 4.1} - \frac{1}{3^5 + 4.3} + \frac{1}{5^5 + 4.5} - \ldots$$

Nilakantha also formulated following identities :

$$\sin(x+\delta x)-\sin x = 2\sin\left(\frac{\delta x}{2}\right)\cos\left(x+\frac{\delta x}{2}\right)$$

$$\cos(x+\delta x)-\cos x = -2\sin\left(\frac{\delta x}{2}\right)\sin\left(x+\frac{\delta x}{2}\right)$$

He also expressed that $\sin(x+\delta x)-\sin x$ varies as the value of cosine varies and corresponding to increase of the value of $x$, its value gets decreased. Similarly, the value of $\cos(x+\delta x)-\cos x$ varies as the value of sine varies negatively and when the value of $x$ increases its value also get increased. It is interesting to note here that Nilakantha Somayaji made significant contributions towards the second order finite differences in interpolation, which were studied geometrically by using the properties of the circle and similar triangles. The second difference for sine difference and consine difference he denoted by $\Delta_2$ (sin $x$) and $\Delta_2$ cos ($x$) and explained them as

(*i*) the difference of the sine difference varies as the sine negatively and numerically increases when value of the angle get increased

(*ii*) the difference of the cosine difference varies as the cosine negatively and its value get numerically decreased when the angle get decreased.

Nilakantha gave the formula for the second difference of sine function as

$$\Delta_2 \sin x = -\sin x\left(2\sin\left(\frac{\delta x}{2}\right)\right)^2$$

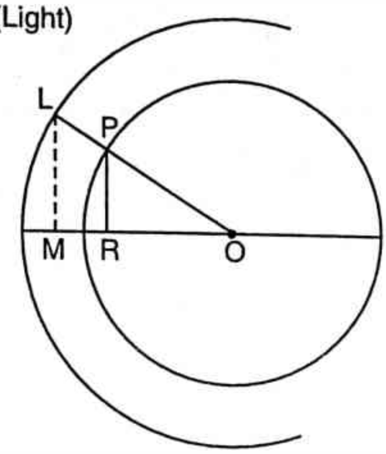
He also used the concept of differential to express the result related to inverse sine function as

$$\delta(\sin^{-1} e\sin\omega) = \frac{e\cos\omega}{\sqrt{1-e^2\sin^2\omega}}\,\delta\omega$$

By using similar triangles, Nilakantha gave the rule for shadow measurement as explained below :

Let source of light L be located on a circle concentric with the circle of gnomon and its shadow, so LM should be sine chord and MO, cosine chord.

From the triangles (similar) LMO and PQO,1



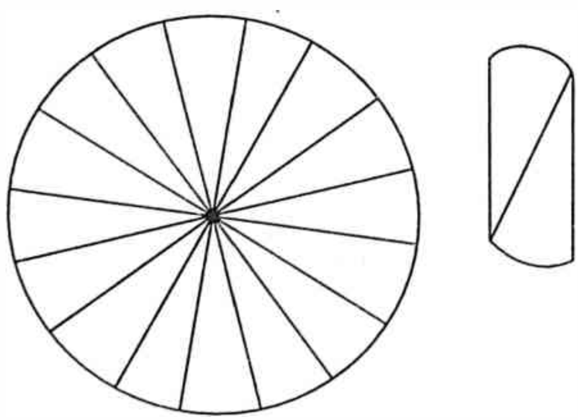

$$\frac{OQ}{OM} = \frac{PQ}{LM}$$

or
$$\frac{OQ}{OM - OQ} = \frac{PQ}{LM - PQ}$$

or
$$\frac{OQ}{MQ} = \frac{PQ}{LM - PQ}$$

and the shadow $OQ = \dfrac{PQ \,.\, MQ}{LM - PQ}$

Nilakantha Somayaji gave a beautiful method to find the area of a circle, in which circle can be divided into large number of Sucyakaraketras by drawing lines from the centre to the circumference, where a small part of Sucyakaraksetras is called as sucis and if these sucis are sufficiently large then the base of these sucis will be straight line and they take the form of triangles and when two of such triangles are combined invertedly then small rectangles emerge with one side equal to the radius of circle and other side equal to the base of sucis (triangles). When these small rectangles are arranged one with other, they will form a big rectangle of one side equal to radius of the circle and other side equal to the half of the circumference of the circle and the area of this big rectangle should be equal to area of given circle *i.e.*



area of circle = area of big rectangle

$$= \left(\frac{1}{2} \text{ circumference of circle}\right) (\text{radius of circle})$$

$$= \pi r \times r = \pi r^2$$

Nilakantha also gave the simple expression to find the value of the arc in terms of chords and the value of common chord and its height in case of intersecting circles.

He discovered the methods to find volume and surface area of a sphere by using the concept of differentiation and integration. For any pyramid with equilateral face he gave the rule to find the Urdhvabhuja (perpendicular height), for which first he found the circumcentre cum orthocentre of the face and its altitude. As it is clear that the line joining the circumcentre to the middle point of the side of the base is one third of its altitude and for any edge of length $a$, altitude is $\sqrt{3a^2/4}$, so the perpendicular height (urdhvabhuja) is $\sqrt{2/3a^2}$ and volume of this type of tetrahedron should be equal to $\sqrt{2/8}\ a^3$, which is quite closer to the correct value of the volume of such tetrahedron as $\sqrt{2/12}\,a^3$. Nilakantha somayaji gave geometrical demonstrations of various algebraic identities. He also explained the method to construct Sredhiksetras (diagrammatic representation of series) to find correct summation of series. Another remarkable contribution of Nilkantha somayaji is related to the convergent geometric series. It seems that he was the first mathematician, who formulated the sum of a convergent infinite geometric series. He explained the sum of an infinite series whose successive terms after the first are obtained by dividing the preceding one by same divisor, remains always equal to the quotient of the first term divided by one less the common divisor.

*e.g.*, $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots$ to infinity its sum to infinite terms should be equal to

$$S_\infty = \frac{1}{1-(1/4)} = \frac{4}{3}.$$

In general it can be expressed in the modern notation that if $a$ is the first term and $r$ is the common ratio, then geometric series should be in the form $a + ar + ar^2 + ar^3 + \ldots$ to infinity.

The sum of this series can be obtained, provided it is convergent, for which the value of common ratio should remain in between $-1$ and 1. *i.e.*, $-1 < r < 1$ and $S_\infty = \frac{a}{1-r}$.

# 21. Ganesa

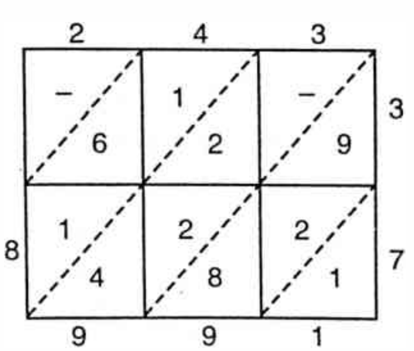
Ganesa was born in 1507 A.D., at Nandigrama. His father Kesava was a famous astronomer. Some historians are of the opinion that birth place of Ganesa was in Konkan region, nearly sixty kilometers away from south of Bombay. Ganesa was fortunate to have the knowledge of astronomy and mathematics from his father Kesava and grand-father Kamalakara. Kamalakara was an illustrious mathematician and astronomer who contributed much in the field of mathematics and astronomy, details about which are being given separately in this book. Kesava wrote several texts related to astronomy and astrology and he was considered best observational astronomer of that period. Kesava taught the basics of astronomy and mathematics to Ganesa and trained him properly for further advancement in that direction. Ganesa also wrote several texts related to mathematics, astronomy and astrology namely, Grahalagheva Laghu and Brihat-Tithi Chintamani, a commentary on Bhaskara II's Siddhanta-Siromani, Budhi-Vilasini, a commentary on Bhaskara II's Lilavati, Vivaha Vrindavana Tika, Muhurta Tattvatika, Sraddha Nimaya, Parva Nimaya and Pata Sarani. Ganesa was a brilliant author, which can be evidently seen in his texts/commentaries. His presentation of various topics was quite unusual and rather interesting as Ganesa expressed the data of composition of Vrindavanatika in the way "take 12 as the number for the samvatsara (hayan) add one for ayana to it, add 6 to the sum of these two numbers, add 4 to the sum which would give 23 as the number of nakastra and one for paksa. If a paksa is added to one more paksa, it would give 3 as the number denoting the week day, take as the tithi number and 11 as the month number, multiply the sum of all these numbers by 21 and increase the product by 9, the result is saka year number". (Tr. by S. Balachandra Rao)

It is calculated as

$$(12 + 1 + 19 + 23 + 1 + 3 + 1 + 11) \times 21 + 9 = 1500,$$

which shows that Vivaha-Vrindavanatika was composed in the Saka year 1500 or 1578 A.D.

In Graha-Laghva, Ganesa made the rule of computations for the positions of planets very simple. He also adopted a new set of ahargana cycle of 4016 days, which is approximately equal to eleven years to avoid large number in calculations. Graha-Laghva, consisting of 187 slokas and arranged into 14 chapters. It is considered an important astronomical text of medieval period, which is quite useful even in modern period to get relevant informations in this area. Many scholars from different parts of India wrote commentaries on this text and its popularity can be understood by this fact that Graha-Laghva is being used even in present times by Panchanga makers in the states like Maharastra, Gujarat, Madhya Pradesh, Karnataka and Deccan region of Andhara Pradesh. Ganesa in his commentaries on the works of Bhaskara II also explained and in some cases improved the results given by his predecessors in the fields of mathematics like arithmetic, algebra, geometry, mensuration etc. He explained a particular kind of multiplication technique and named it as Kapata-Sandhi. It is also called as gelosia multiplication *e.g.*, for the multiplication of 243 by 37, he demonstrated this method in the following way :



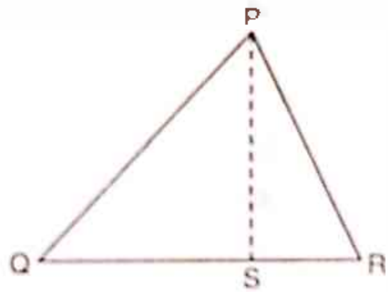

$$\Rightarrow \qquad 243 \times 37 = 8991$$

Ganesa also described another method of multiplication Tastha-Gunana, which is also known by other names like Triyak-Gunana or Vajrabhyasa. It is interesting to mention that Ganesa was very clear about the notion of infinity as he defined the quantity $\left(\frac{a}{o}\right)$ by infinite, unlimited, indefinite, which can not be altered by adding or subtracting any finite quantity to it *i.e.* $\frac{a}{o} \pm p = \frac{a}{o}$.

Ganesa solved indeterminate equations of first degree like $by - ax = \pm c$. The analysis of these type of equations was in general called by Hindu mathematicians Kuttaka analysis. Ganesa explained the meaning of Kuttaka as a term for the multiplier. In fact, these type of indeterminate equations were needed to solve various astronomical problems and Ganesa stated that astronomical problems involving the equation $by - ax = \pm c$ have the condition that dividend ($a$) and divisor ($b$) are constant but the interpolator ($c$) always varies. The particular form of these equations *i.e.* $by - ax = \pm 1$, was called by him as Sthira-Kuttka (constant pulverizer).

Ganesa gave the complete geometrical proof of Sulba or Baudhayana theorem in following way :

Let PQR be a triangle right angled at P. Draw PS perpendicular to QR, then triangles PSQ, RSP and RPQ are similar. From the similar triangles PSQ and RPQ, we get the ratios of sides as




$$\frac{PS}{RP} = \frac{SQ}{PQ} = \frac{PQ}{RQ}$$

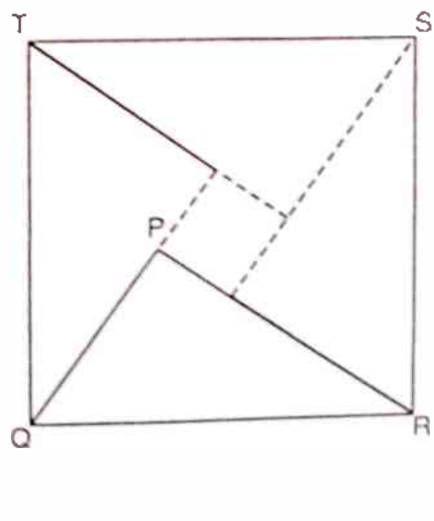
$$\Rightarrow \qquad SQ = \frac{PQ^2}{RQ}$$

Similarly, from the similar triangles RSP and RPQ, we get,

$$\frac{RS}{RP} = \frac{SP}{PQ} = \frac{RP}{RQ} \quad \text{or} \quad RS = \frac{RP^2}{RQ}$$

$$\therefore \quad SQ + SR = \frac{PQ^2 + RP^2}{RQ} \quad \text{or} \quad QR^2 = PQ^2 + RP^2$$

This result was also proved by Ganesa in other way, taking into consideration the explanation of Bhaskara II in this regard *i.e.* the sum of the squares of the bhuj and koti will be equal to twice the product of bhuj and koti combined with the square of their difference. Its proof is given below. Let PQR be a triangle, right angled at P. The hypotenuse QR is the side of square QRST. Now considering the sides SR, ST and TQ as



hypotenuse three other right triangles are drawn equal and similar to the triangle PQR, forming a small square at the centre with the sides equal to the difference of buja and koti. Then the area of each triangle $= \frac{1}{2} \times$ bhuja $\times$ koti.

So, area of all four triangles

$$= 4 \times \frac{1}{2} \times \text{bhuja} \times \text{koti} = 2\ \text{bhuja} \times \text{koti}$$

The area of bigger square

$$= \text{area of all four triangles}$$
$$+ \text{area of small square formed at the centre}$$
$$= 2\ \text{bhuja} \times \text{koti} + (\text{bhuja} - \text{koti})^2$$
$$= 2\ \text{bhuja} \times \text{koti} + \text{bhuja}^2 + \text{koti}^2 - 2\ \text{bhuja} \times \text{koti}$$
$$= \text{bhuja}^2 + \text{koti}^2$$

which gives the result in the form

$$QR^2 = QP^2 + PR^2$$

In case of rational right angled triangle Ganesa expressed that if the upright, base and hypotenuse of any rational right angled triangle are multiplied by any arbitrary rational number then it will produce another rational right angled triangle. On the basis of this assumption, Ganesa gave the general solution of $x^2 + y^2 = z^2$ in integers as $(p^2 - q^2)\, r$, $2pqr$, $(p^2 + q^2)\, r$, where $p$, $q$ and $r$ are rationals. Ganesa also described a method for the construction of rational quadrilateral by forming four rational right triangles from two basic right triangles. According to Ganesa if $a^2 - b^2$, $2ab$, $a^2 + b^2$ and $c^2 - d^2$, $2cd$, $c^2 + d^2$ are two rational right angled triangles then the following four rational right triangles are formed by these two right angled triangles and they together form the rational quadrilateral. The sides of those four rational right triangles should be in following combinations :

(*i*) $(a^2 - b^2)(c^2 - d^2)$, $2ab\,(c^2 - d^2)$, $(a^2 + b^2)(c^2 + d^2)$

(*ii*) $(a^2 - b^2)(2cd)$, $4abcd$, $(a^2 + b^2)(2cd)$

or $2(a^2 - b^2)(cd)$, $4abcd$, $2(a^2 + b^2)\, cd$

(*iii*) $(c^2 - d^2)(a^2 - b^2)$, $2cd\,(a^2 - b^2)$, $(c^2 + d^2)(a^2 - b^2)$

(*iv*) $(c^2 - d^2)(2ab)$, $2cd\,(2ab)$, $(c^2 + d^2)(2ab)$

or $2ab\,(c^2 - d^2)$, $4abcd$, $2ab\,(c^2 + d^2)$

Ganesa gave a beautiful method to find the area of a circle by cutting it into large number of sucyakaraksetras. In this method first circle is to be cut into two semicircles and then same number

of radial lines are cut in both the halves and later on joined invertedly to produce a big rectangle with length equal to half of the circumference of given circle and breadth, the radius of circle making area of resultant rectangle equal to the area of given circle

*i.e.*, area of circle = area of rectangle

$$= \frac{1}{2} \times \text{circumference} \times \text{radius}$$

$$= \pi r^2$$

He also described a method to inscribe a polygon in a circle, for which he divided the circumference of a circle into as many equal parts as desired and then calculated the value of the chord corresponding to one division by using sine table. He also gave one particular example to calculate the side of any regular polygon inscribed in a circle by taking the sides of polygon equal to 384 and diameter of circle 100, for this, he calculated the value of side by using the formula

$$S_{2n} = \sqrt{\frac{S_n^2}{4} + \left(r - \sqrt{\frac{4r^2 - S_n^2}{4}}\right)^2}$$

where $S_{2n}$ and $S_n$ are the sides of the inscribed polygons of sides $2n$ and $n$ respectively. With the application of this formula it can easily be seen that the side of any regular hexagon inscribed in a circle should be equal to the radius of the circle. With the help of this notion, Ganesa also found the approximate value of $\pi$ as $\frac{3927}{1250}$ which is equal to 3.1416 ...

Ganesa also demonstrated a method to find the volume of obelisk as the sum of the volume of the prism at the centre, volume of four pyramids at the corners and volume of four prims on four sides. If $(x, y)$ and $(x', y')$ denote the length and breadth of the face and base of the solid respectively and $h$, the height, then volume of obelisk should be equal to $\frac{h}{6}\{(x + x')(y + y')\ xy + x'y'\}$

In general it can be noted that Buddhivilasini, a commentary of Ganesa on the famous mathematical treatise Lilavati of Bhaskaracharya II is an interesting and useful mathematical text to understand various rules, theorems, proofs, expositions related to mathematics.

# 22. Jyesthadeva

Jyesthadeva was born probably in 1500 A.D. in the Parannottu family of Alattur, a place situated in the South Malabar region of Kerala. He was a student of Damodar of Vatasreni and later on became an important member of Kerala school of mathematics which was formed on the mathematical works of Parameswara, Nilakantha, Madhava and many other scholars of mathematics and astronomy of that region. Jyesthadeva wrote a very important mathematical text named as Yuktibhasa or Ganita-Nyaya-Sangrah, which incorporated almost all the theorems and formulae which were in use during his period among the astronomers and mathematicians of Kerala and presented the rationale and proof of those theorems in this text. According to the scholar K.V. Sharma,"he is no innovator in the line of derivation and proof." Yuktibhasa presented systematic expositions of these mathematical theorems, formulae and results and covered the topics related to mathematics in the first part of Yuktibhasa and the topics related to astronomy in its second part. Jyesthadeva originally wrote Yuktibhasa in Malayalam but its Sanskrit version is also available now with the name "Ganita—Yuktibhasa". The astronomical part of Yuktibasa covers the topics related to epicyclic and eccentric models of planetry motions, sun's position on the celestial sphere and its relationships with three systems of coordinates *i.e.* ecliptic, equatorial and horizontal coordinates ; lunar and solar eclipses, deviation of the longitudes of the sun and the moon, rising and setting of the moon and planets and size of the moon which is illuminated by the sun and graphical representation of this illuminated size and the part of moon.

The mathematical part of Yuktibhasa is very important in terms of its style of presentation and coverage of all important topics discovered by his predecessors. In fact, Yukibhasa has become an important source to get the references of all those topics which were explained and discovered by famous mathematician of Kerala of

fourteenth century, *i.e.*, Madhava of Sangamagarama. This part of Yuktibhasa dealt with the problems related to Geometry, Algebra and Arithmetic. In Geometry he explained the properties related to plane figures *viz.* triangle, quadrilateral, circle etc. Jyasthadeva explained the properties of triangles as

(*i*) altitude of an isosceles triangle bisects its base

(*ii*) altitude of a scalene triangle is always nearer to the shorter side

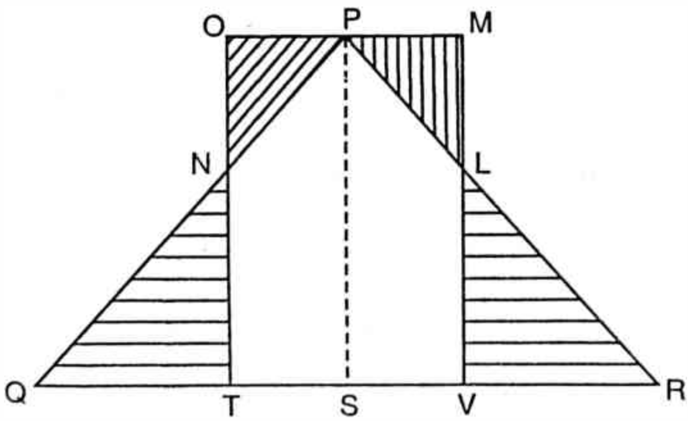
(*iii*) if one side and a hypotenuse of one triangle are respectively perpendicular and parallel to hypotenuse and one side of other triangle, then these two right angled triangles are similar

(*iv*) if all three sides of one triangle are parallel to the three sides of the other triangle, then these right triangles are similar

(*v*) if three sides of one right triangle are perpendicular to the three sides of the other right triangle, then they are similar

(*vi*) in similar figures the ratios of corresponding sides are equal.

Jyesthadeva presented a very interesting proof to find the area of any triangle equal to half of the product of base and altitude. To prove this, he followed this method :




Let PQR be the triangle, PS be the altitude, T and U be the mid-points of abadhas QS and SR and N, L be the midpoints of sides PQ and PR take triangles NTQ and LVR out from the triangle and place them upward in such a way that QN coincides with NP and LR coincides with LP, which finally form the rectangle TVMO, whose length is equal to altitude of triangle and breadth

$TV = TS + SV = \frac{1}{2} QS + \frac{1}{2} SR$, $TV = \frac{1}{2} QR$. Therefore the area of

rectangle which is also equal in area to given triangle is equal to $\frac{1}{2}$ QR × PS. *i.e.* $\frac{1}{2}$ × base × altitude.

He also gave another rule to find the altitude of triangle, which should be equal to the product of two sides (other than the base) divided by the diameter of the circumcircle. In Yuktibhasa we also find a beautiful geometrical demonstration of Baudhayana theorem related to right angled triangle. To find the value of $\pi$ equal to square root of 10 he followed the method of the escribed polygon starting from the square and proceeding further to the polygon of large number of sides. In Yuktibhasa, there is a statement for getting the value of $\pi$ in the form of infinite series by using the method of integration. This method is originally attributed to Tantrasangraha and is derived from Madava's expositions. It is expressed in following verse as,

व्यासे वारिधिनिहिते रूपहृते व्याससागरभिहते

त्रिशरादि विषम संख्या भक्तं ऋणं स्वं पृथक्क्रमात् कुर्यात

(Tantrasamgraha quotation in the Yuktibhasa p. 99)

According to this sloka if diameter is multiplied by 4 and divided by one, further again diameter multiplied by 4 and divided by odd numbers 3, 5, etc. and alternatively added and subtracted then value of circumference is obtained *i.e.*

$$\text{Circumference} = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \ldots\ldots$$

or $$\frac{\text{Circumference}}{d} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \ldots\ldots$$

or $$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \ldots\ldots$$

Yuktisbhasa also dealt with cyclic quadrilaterals in detail and presented many geometrical results by applying trignometrical identities. We find following trigonometric identities in Yuktibhasa :

$$\sin^2 A - \sin^2 B = \sin(A + B) \sin (A - B)$$

$$\sin A \,.\, \sin B = \sin^2 \left(\frac{A + B}{2}\right) - \sin^2 \left(\frac{A - B}{2}\right)$$

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$d(r \sin A) = \frac{r \cos A \,.\, dA}{r}$$

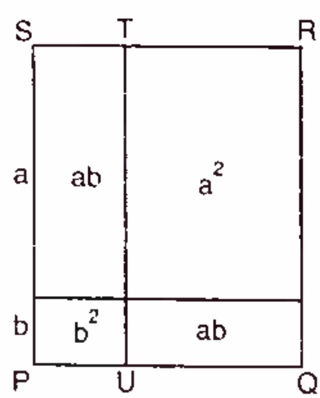
$$d(r\cos A) = \frac{r\sin A \,.\, dA}{r}$$

Jyesthadeva presented various mathematical theorems and results discovered by Madhava and Nilakantha with proofs. The series of arc (tan $x$), first discovered by Madhava got beautiful exposition in the Yuktibasa as "the first term is the product of the given sine and radius of the desired arc divided by the cosine of the arc. The succeeding terms are obtained by a process of iteration when the first term is repeatedly multiplied by the square of the sine and divided by the square of the cosine. All the terms are then divided by the odd numbers 1, 3, 5, ..... . The arc is obtained by adding and subtracting respectively the terms of odd rank and those of even rank. It is laid down that the sine of the arc or that of its complement whichever is smaller should be taken here as the given sine. Otherwise, the terms obtained by this above iteration will not tend to the vanishing magnitude"

[Translated by R.C. Gupta]

It is interesting to note here that this remarkable exposition about Madhava's series related to arc (tan $x$) was given by Jyesthadava almost hundered years before the Gregory's description about it, *i.e.* in 1671 A.D. in Europe. The sine and cosine series discovered by Madhava were also explained and proved in Yuktibhasa. Again it will be appropriate to mention here that the expansion of trigonometric functions in the form of infinite series were known to ancient Hindu mathematicians even in fourteenth century whereas the same concept became known to Europeans only in seventeenth century. In Yuktibhasa we also see interesting demonstration of algebraic geometery. Few of these geometric demonstrations are given below :

(*i*) $(a + b)^2 = a^2 + b^2 + 2ab$



(*ii*) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, this algebraic identity is demonstrated in following figure

| | a | b | c |
|---|---|---|---|
| a | $a^2$ | ab | ac |
| b | ba | $b^2$ | bc |
| c | ca | cb | $c^2$ |

Jyesthadeva also gave geometrical representation for the multiplication of fractions and square of any number $n$ expressible as the sum of the series 1, 3, 5 ... upto $n$ terms, which was also called as Srediksetra *i.e.*, diagrammatical representation of series. Jyesthadeva also used the concepts of differentiation and integration to find the volume and surface area of a sphere, which are very similar to the modern methods of finding volume and surface area of sphere by using integration and propounded by Newton and Lebinitz. Yuktibhasa is also known as the first calculus text of the world. In this text Jayasthadeva described the summary of contributions of Kerala school of mathematicians particularly in the areas of calculus, trigonometry and mathematical analysis, most of which were attributed to illustrious mathematican Madhava of 14th century. The important developments and discoveries incorporated in this book include the power series, expansions of the functions in the form of infinite series, trigonometric series of sine, cosine, tangent and arc tangent, tests of convergence, power series of $\pi$, the derivative numerical integration by means of infinite series, the relationship between the area of a curve and its integral, the mean value theorem etc. The method of integration to find the volume of sphere is explained below.

Divide the sphere into large number of strips by drawing circles parallel to horizontal great circle at equal distances and then cut it into circular laminae by planes with uniform thickness equal to one unit.

Volume of a lamina $= \pi c^2 \,.\, 1$, where $c$ is the radius of the circle and is equal to its corresponding sine chord.

The volume of sphere should be equal to sum of the areas of al circular laminae multiplied with thickness *i.e.*

$$V = \pi(c_1^2 + c_2^2 + ..... + c_n^2) . 1$$

Further, if $k$ is the height of double the arc sine chord is $c$ and $d$ the diameter, then

$$c^2 = k(d-k) = \frac{2k(d-k)}{2}$$

$$= \frac{d^2 - \{k^2 + (d-k)^2\}}{2}$$

Therefore,

$$c_1^2 + c_2^2 + ..... + c_n^2$$

$$= \frac{d^2 - \{k_1^2 + (d-k_1)^2\}}{2} + \frac{d^2 - \{k_2^2 + (d-k_2)^2\}}{2}$$

$$+ ..... + \frac{d^2 - \{k_n^2 + (d-k_n)^2\}}{2}$$

$$= \frac{n}{2}d^2 - \frac{\Sigma k_n^2}{2} - \frac{\Sigma (d-k_n)^2}{2}$$

If sphere is divided into large number of laminae then its thickness will be infinitely small and $k_1 = 2\Delta d$ ..... $k_n = n\Delta d$

and $\quad \Sigma k_n = \Sigma(d - k_n)$

and $\quad \Sigma k_n^2 = (\Delta d)^2 + (2\Delta d)^2 + ... + (n\ \Delta d)^2 = \dfrac{d^3}{3}$

because $n$ also becomes in such a case equal to the diameter of the circle

$$\Sigma c_n^2 = \frac{d^3}{2} - \frac{d^3}{2} = \frac{d^3}{6}$$

So value of sphere $= \pi\Sigma c_n^2 . 1 = \dfrac{\pi . d^3}{6} = \dfrac{4}{3}\pi r^3.$

# 23. Kamalakara

Kamalakara was born in the year 1616 A.D. in Benares. His father was Narsimha, who was also an astronomer. Kamalakara's elder brother Divakara and younger brother Rangnatha were also famous mathematicians and astronomers. Kamalakara got his basic education in mathematics and astronomy from his elder brother Divakara. His most celebrated text is "Siddhanta-Tattva-Viveka", which was completed in the year 1658 A.D. His other works are : Grahagolatattva, Grahasarani, Kairasyudaha-rana, a commentary on the Bhaskaracharya's Lilavati, Manorama, a commentary on the Grahalaghava of Ganesa Daivajna, Sauravasana, a commentary on the Suryasiddhanta and Sesavasana, a supplement to the Siddhantatattvaviveka. In Siddhantatattvaviveka, Kamalakara mentioned that his ancestors belonged to the village Golagrama which was situated near the river Godavari.

Kamalakara's text Siddhanta-Tattva-Viveka consists of fifteen chapters containing almost all important topics related to astronomy like mean motions of planets, true longitudes of the planets, diameters and distances of the planets, the earth's shadow', the moon's crescent, the patas of the moon and the sun, the 'great problem', planetary transits across the sun's disk. Kamalakara wrote Siddhanta-Tattva-Viveka in Sanskrit and very frequently used the place value number system with sanskrit numerals. The Sanskrit names for 1 to 9 used in this text were : eka, dvi, tri, chatur, pancha, shat, sapta, ashta and nava. These numbers were also denoted by different words of symbolical value *e.g.* the words like eka, pitamaha, adi .... were used for one ; dvi, yama, ashvin, bahu, paksha ..... were used for two ; tri, guna, kala, agni ... were used for two ; tri, guna, kala, agni ... were used for three ; chatur, sindu, yuga, brahmasya, ... were used for four ; pancha, bana, indriya, bhuta, mahayajna ... were used for five ; shat, rasa, anga, shanmukha were used for six ; sapta, ashva, svara, sagara ... were

used for seven ; ashta, gaza, naga, murti ... were used for eight ; nava, anka, graha ... were used for nine and shunya, bindu, kha, akasha, ambara ... were used for zero.

The third chapter of Siddhanta-Tattva-Viveka deals with trigonometric functions. He used addition and subtraction theorem for sine and cosine to get the values of sines and cosines of doubles, triple, quadruple and quintuple angles. Kamalakara formulated following results :

$$\sqrt{R^2 - (\text{jya }\alpha)^2} = \text{kojya }\alpha$$

$$\sqrt{R^2 - (\text{kojya }\alpha)^2} = \text{jya }\alpha$$

$$R - \text{jya }\alpha = \text{utjya }(90° - \alpha)$$

$$R - \text{jya }(90° - \alpha) = \text{utjya }\alpha$$

Kamalakara beautifully expressed that if R sine of an arc multiplied with radius is added to or subtracted from the square of radius and further square root of half of this value is taken, then it will give the value of the R sine of the half of three signs plus or minus the arc *i.e.*

$$\text{jya }\frac{1}{2}(90° \pm \alpha) = \sqrt{\frac{1}{2}(R^2 \pm R\text{ jya }(90° - \alpha))}$$

this formula can be transformed to the modern form as

$$2\cos^2\left(\frac{\theta}{2}\right) = 1 + \cos\theta \;;\; 2\sin^2\left(\frac{\theta}{2}\right) = 1 - \cos\theta$$

He further explained the rule of find the sum of difference of sine and cosine as "the quotients of the R sines of any two arcs of a circle divided by its radius arc reciprocally multiplied by their R cosine ; the sum and difference of them are equal to the R sine of the sum and difference respectively of the two arcs".

(Translated by Datta & Singh)

*i.e.* $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

Similarly, "the product of the R cosines and of the R sines of two arcs of a circle are divided by its radius ; the difference and sum of them (the quotients) are equal to the R cosine of the sum and difference of the two arcs"

(Translated by Datta & Singh)

*i.e.* $\text{kojya }(\alpha \pm \beta) = \left(\frac{\text{kojya }\alpha\,.\,\text{kojya }\beta}{R}\right) \mp \left(\frac{\text{jya }\alpha\,.\,\text{jya }\beta}{R}\right)$

which is equivalent to

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

For these theorems Kamalakara also made it clear that when two arcs go beyond 90° to any even or odd quadrant, in such cases also the above theorems remain same. This explanation shows that Kamalakara gave the general proof of these theorems considering all possible values of the arcs. It is interesting to see that Kamalakara had extensively considered different cases for all integral and fractional values of the arc. For double, triple, quadruple and quintuple angles he stated that

(*i*) if R sine of an arc is multiplied with R cosine of same arc, further multiplied by two and divided by radius then we get R sine of twice of that arc *i.e.*

$$\text{jya } 2\alpha = \frac{2\text{jya } \alpha \text{ kojya } \alpha}{R}$$

or $$\sin 2\theta = 2 \sin\theta \cos\theta$$

(*ii*) if the square of R sine of an arc is subtracted from the square of R cosine of same arc and the difference thus obtained is divided by the radius, then we get R cosine of twice of that arc *i.e.*,

$$\text{kojya } 2\alpha = \frac{(\text{kojya } \alpha)^2 - (\text{jya } \alpha)^2}{R}$$

or $$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Similarly, using the addition theorem successively, Kamalakara obtained following results :

$$\text{jya } 3\alpha = \{3R^2 \text{ jya } \alpha - 4(\text{jya } \alpha)^3\}/R^2$$

$$\text{kojay } 3\alpha = \{4(\text{kojya } \alpha)^3 - 3R^2 \text{ kojya } \alpha\}/R^2$$

$$\text{jya } 4\alpha = 4\{(\text{kojya } \alpha)^3 \text{ jya } \alpha - (\text{jya } \alpha)^3 \text{ kojya } \alpha\}/R^3$$

$$\text{kojya } 4\alpha = \{(\text{kojya } \alpha)^4 - 6(\text{kojya } \alpha)^2 (\text{jya } \alpha)^2 + (\text{jya } \alpha)^4\}/R^3$$

$$\text{jya } (5\alpha) = \{(\text{jya } \alpha)^5 - 10(\text{jya } \alpha)^3 (\text{kojya } \alpha)^2 + 5(\text{jya } \alpha)(\text{kojya } \alpha)^4\}/R^4$$

$$\text{kojya } (5\alpha) = \{(\text{kojya } \alpha)^5 - 10(\text{kojya } \alpha)^3 (\text{jya } \alpha)^2 + 5(\text{kojya } \alpha)(\text{jya } \alpha)^4\}/R^4$$

These formulas are equivalent to the following trigonometric identities, which are expressed in modern form as

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\sin 4\theta = 4(\cos^3\theta \sin\theta - \sin^3\theta\cos\theta)$$
$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$\sin 5\theta = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta$$
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

Kamalakara also established the formulae for jya function o the half, third, fourth and fifth part of an arc and further suggestec that the values of other submultiplies of an arc can also be calculated in the similar way,

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1}{2}(1-\cos\theta)}$$

$$\text{jya}\left(\frac{\alpha}{3}\right) = \frac{1}{3}\text{jya}\,\alpha + \frac{4}{3R^2}\left(\frac{\text{jya}\,\alpha}{3}\right)^3$$

or $$\sin\left(\frac{\theta}{3}\right) = \frac{1}{3}\sin\theta + \frac{4}{81}\sin^3\theta$$

$$\text{jya}\left(\frac{\alpha}{4}\right) = \frac{1}{2}\sqrt{2R^2 - \frac{R^2(\text{jya}\,\alpha)}{\text{jya}(\alpha/2)}}$$

or $$\sin\left(\frac{\theta}{4}\right) = \frac{1}{2}\sqrt{2 - \frac{\sin\theta}{\sin(\theta/2)}}$$

$$\text{jya}\left(\frac{\alpha}{5}\right) = \frac{1}{5}\text{jya}\,\alpha + \frac{4}{R^2}\left(\frac{\text{jya}\,\alpha}{5}\right)^3 - \frac{16}{5R^4}\left(\frac{\text{jya}\,\alpha}{5}\right)^5$$

or $$\sin\left(\frac{\theta}{5}\right) = \frac{1}{5}\sin\theta + 4\left(\frac{\sin\theta}{5}\right)^3 - \frac{16}{5}\left(\frac{\sin\theta}{5}\right)^5$$

He subsequently explained the method to calculate the jya function for half of the difference of two arcs, for which he stated that by taking the sum of the squares of the differences of R sines and R cosines of two arcs and after taking the square root of this sum, making it half, we get the required value *i.e.*

$$\text{jya}\,\frac{1}{2}(\alpha-\beta) = \frac{1}{2}\{(\text{jya}\,\alpha - \text{jya}\,\beta)^2 + (\text{kojya}\,\alpha - \text{kojya}\,\beta)^2\}^{1/2}$$

or $$\sin\frac{1}{2}(\theta-\phi) = \frac{1}{2}\{(\sin\theta - \sin\phi)^2 + (\cos\theta - \cos\phi)^2\}^{1/2}$$

It is also interesting to mention here that Kamalakara gave beautiful methods to find the value of jya 18 and jya 36. He denoted

the value of jya 18 by $y$ and radius by R and established the relations as :

$$\frac{1}{2}R(R-y) = \frac{1}{2}R \text{ . utjya } 72 = (\text{jya } 36°)^2$$

and $$\frac{2y^2}{R} = \text{utjya } 36°$$

From these two equations, we get

$$\frac{1}{2}R(R-y) + \left(\frac{2y^2}{R}\right)^2 = (\text{jya } 36°)^2 + (\text{utjya } 36°)^2$$

or $$\frac{1}{2}R(R-y) + \left(\frac{2y^2}{R}\right)^2 = 4y^2$$

Solving this equation for $y$, we get

$$y = \frac{1}{4}(\sqrt{5R^2} - R^2) = \text{jya } 18°$$

and thereafter,

$$\text{jya } 36° = \sqrt{\frac{1}{8}(5R^2 - \sqrt{5R^4})}$$

Kamalakara also solved problems related to algebra. He dealt with the indeterminate equations of first degree like $by - ax = \pm c$. He also gave the solutions for indeterminate quadratic equations like $Nx^2 + 1 = y^2$, which were generally called the equations of Varga-Prakrti or Krti-Prakrti. For these type of equations he stated that if $x = p$ and $y = q$ be a solution for $Nx^2 + c = y^2$, then $p < q$, if N and C are positive but if N and C are of opposite signs then for some cases $p$ may be greater than $q$. For the equation $Nx^2 + 1 = y^2$, he expressed that if $l$ be an optional number, then $\frac{2l}{l^2 - N}$ should be lesser root of $Nx^2 + 1 = y^2$. Substituting this value of $x$ in the equation, we get

$$y^2 = N\left(\frac{2l}{l^2 - N}\right)^2 + 1 = \left(\frac{l^2 + N}{l^2 - N}\right)^2$$

so, greater root for this equation should be $y = \frac{l^2 + N}{l^2 \sim N}$. Kamalakara also solved the equations of the form $ax^2 + by^2 + c = z^2$, considering different possibilities of signs and values for constants. *e.g.* for the equation $ax^2 + by^2 + c = z^2$, where $c$ is in square form say $d^2$, Kamalakara gave the solution as $x = \frac{b}{2ad}\,y^2$ and $z = \frac{by^2}{2d} + d$.

# 24. Srinivasa Ramanujan (1887 A.D.–1920 A.D.)

Srinivasa Aiyangar Ramanujan Aiyangar was born on December 22, 1887 in the house of his maternal grandfather at Erode in Tamilnadu. There is a legend attached to the birth of Ramanujan that for many years after the marriage, when his parents could not get the birth of child then his mother and maternal grandfather held special prayers in Namakkal to please the goddess Namagiri. Their prayers were answered by the birth of a male child, who later on became famous as a genius of mathematics throughout the world. Ramanujan also developed same sense of devotion as his mother, to Goddess Namagiri from childhood itself and very often experienced that various formulae, theorems, conjectures and mathematical problems came to his mind in the dreams for which he was inspired by the Goddess herself. Ramanujan belonged to Vaishnava Brahmin family of Kumbakonam, a place famous for its temples. His paternal house was also very nearer to the famous Saranga Pani temple. His father was a petty accountant in the shop of a local cloth merchant and his economic condition was not good. From infancy itself Ramanujan was quiet, reticent and meditative in nature and always liked to remain in solitude. As his mother was highly religious in nature so Ramanujan also learnt slokas, hymns and other sacred religious songs from his mother and gradually developed his interest towards religious texts, epics and other literature related to Vedic lore. He also learnt Sanskrit and Tamil and got mastery in reciting the slokas and hymns from the Vedas, Upanishads, Tirukkural and other important scriptures written in Tamil and Sanskrit. At the age of five, he was enrolled to the local primary school (piol) at Kumbakonam, from where his formal education in the areas of basic arithmetic and Tamil language started. In school, sometimes his teachers and classmates would become awestruck by seeing the specificity of the mind of Ramanujan in such an early stage and his subtler questions related to stars, their distances and size. In 1894, Ramanujan was admitted

to the Town High Schol at Kumbakonam and from this school he passed Primary Examination in 1897 with compulsory subjects Tamil and Arithmetic and optinal subjects Geography and English. He stood first among all the successful candidates in whole Tanjore district, which made him eligible for half fee concession in the higher classes. In middle classes he took Sanskrit as his second language. He always remained engrossed with some problems of mathematics and while he was only in second form, he started thinking about the highest truth of mathematics. He even asked the question "what is highest truth in mathematics" to the students of higheer classes, for which some replied "Theorem of Pythagoras" is the highest truth, whereas, few other gave "Stocks and Shares" the highest place but he was not satisfied with these answers. When he was in third form, his teacher explained a problem related to division in the way that "if three bananas are distributed among three people, each one will get one banana, if thousand bananas are distributed among thousand people, then also each one will get one banana, which makes it clear that if any quantity is divided by itself, the value obtained will be always one". With this proposition, Ramanujan immediately asked the question : "Sir, what value we get if zero is divided by itself". This question of Ramanujan is sufficient to understand that from such an early age he was quite clear about the properties of numbers alongwith their behaviour with various arithmetical operations. In fact, when he was a student of third form, he was able to solve the problems related to arithmetic, geometric and harmonic progressions. In fourth form itself, he solved all the problems given in Loney's Trigonometry and recognised the properties of trignometric functions that they are not merely the ratios of the sides of any right angled triangle but can also be expressed in the forms of infinite series. He also expressed the values of other transcendental numbers like 'pi' and '*e*' to several places of decimals. He also obtained the results for sine and cosine similar to the Euler's Theorem and explained the methods to solve cubic and quartic equations. During his period of study in Town High School, he came across the book of George Shoobridge Car "A Synopsis of Elementary Results in Pure Mathematics", which contained nearly 6165 formulae, theorems and short proofs related to Algebra, Geometry, Trigonometry and Calculus. Ramanujan mastered almost all the concepts given in this text. Regarding the importance of Car's Synopsis Prof. Hardy later on slated that" this book is not in any sense a great one but Ramanujan has made it famous and there is no doubt that it influenced him profoundly .....".

In 1903, Ramanujan passed Matriculation Examination with first class. He joined Government college Kumbakonam for the F.A. class with subjects English, Mathematics, Physiology, Roman and Greek History and Sanskrit, but as he was growing in age, he was getting deeply engaged with his mathematical sojourns. He found the series $\sum \frac{1}{n}$ and computed the value of Euler's constant upto 15 decimal places and independently started studying the Bernoulli numbers. He was devoting most of his time towards mathematics and neglected other subjects, due to which he failed in F.A. examination which deprived him from the scholarship. Thereafter, it became very difficult for him to continue his studies without scholarship. Facing such brazen realities of poverty, he left the house without the premission of his parents and went to Vizagapatnam and continued his mathematical works on hypergeometric series and relation between integral and series. But again because of some unhappy experiences he returned to home and later on shifted to Madras to continue his studies at Pachaiyappa's college ; where he also got half-free scholarship because of the Principal's positive attitude towards his mathematical interests. But he could not get success in F.A. examination due to illness and returned home. Prof. S.R. Ranganathan described this period of Ramanujan's life as "the depression due to failure in the F.A. examination could not repress it. Failure to get employed could not shake it. Poverty and penury could not obstruct it. His research marched on underterred by any environmental factors physical, personal, economic and social. Magic squares, Continued fractions, Hyper-Geometric series, Properties of numbers—prime as well as composite, Partition of numbers, Elliptic integrals and several other such regions of mathematics engaged his thought. But during, or earlier than that time, hardly any thought was created in the country on some of these problems. The thought created in the West had not even been disseminated in the country. Everything had to be done and discovered by him De-Novo. Ramanujan had cultivated an unusual systematic habit. Each result that he obtained he recorded in a quarto-notebook. Proofs were often absent".

As Ramanujan was doing his mathematical work at home, his parents were getting highly disappointed day after day, so they decided to marry him off to divert his interest from mathematics. In 1909, at the age of twenty two years he was married with eleven years old girl Janaki Devi. After marriage Ramanujan tried to get himself employed somewhere but nothing came easily in his hands

according to his expectations. He moved from one place to other to get some support and lately though, with the influence of some prominent people who also had some interest in mathematics like Prof. Ramaswami Aiyar, Deputy Collector of Tirukoilur, Prof. Seshu Aiyar of Presidency College Madras, Dewan Bhadur R. Ramachandra Rao, the Collector of Nellore District and S. Narayana Iyer of Madras Port Trust he could get the job of a clerk in the Madras Port Trust and joined duty on March 1, 1912. During this period also, he continued his unstinted interests in discovering and investigating the problems of mathematics and got number of papers published in the journal of the Indian Mathematical society. On the advice of Prof. Seshu Aiyar, he sent few of his results related to divergent series to Prof. G.H. Hardy of Trinity College Cambridge on Jan 16, 1913 with a letter stating that "I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only twenty pounds per annum. I am now about 23 years of age. I have had no university education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at mathematics. I have not trodden through the conventional regular course which is followed in a university course but I am striking out a new path for myself. I have made a special investigation of divergent serries in general and the results I get are termed by local mathematicians as startling. Being poor, if you are convinced that there is anything of value I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get but I have indicated to the lines on which I proceed ..... ."

Prof. Hardy studied the theorems and results sent by Ramanujan with Prof. Littlewood and replied on Feb 8, 1913 that "I was exceedingly interested by your letter and by the theorems which you state. You will however understand that before I can judge properly of the value of what you have done, it is essential that I should see proofs of some of your assertions. Your results seem to me to fall into roughly three classes.

(1) there are a number of results that are already known or easily deducible from known theorems
(2) there are results which, so far as I know, are new and interesting but interesting rather from their curiosity and apparent differently than their importance
(3) there are results which appear to be new and important .....

Ramanujan was highly encouraged by the letter of Prof. Hardy and replied him back for help and guidance. He stated that "I have found sympathetically ..... I am already a half starving man. To preserve my brains I want food and this is my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from the Government ....." Though many scholars and officers from Madras and other neighbouring areas, who themselves were interested in mathematics, were also trying genuienly to arrange a constant source of financial help for Ramanujan, so that he could continue his mathematical works without bothering much for his daily livelihood and finally these efforts got succeeded in March, 1913 when University of Madras awarded scholarship of Rs. 75 per mensem to S. Ramanujam for a period of two years with a condition that Ramanujan should submit the quarterly reports of his mathematical work to the University but this scholarship he could avail only for one year because on 17th March 1914, with the great interest and persuastion of Prof. Hardy, S. Ramanujan sailed from India for his academic voyage to Cambridge by the ship. S.S. Nevasa. University of Madras further approved two hundred and fifty pounds per annum scholarship to him for a period of two years from April 1, 1914 with a free passage to and fro from England alongwith petty grant of consolidated amount for other related expenditures. Later on this scholarship was further extended upto five years. On 14th April 1914 he arrived in London and from there he went Cambridge and stayed with Prof. Neville for few weeks until he was alloted accomodation in Trinity college. Soon after his arrival in Cambridge Prof. Hardy entrusted the responsibility of teaching the essentials of modern mathematics needed for Ramanujan's further advancement in research to Prof. Littlewood, but unfortunately, after some time war broke out and Prof. Littlewood was deputed for war duty. Thereafter, only Prof. G.H. Hardy was in Cambridge to help and guide Ramanujan. Prof. Hardy treated S. Ramanujan as his talented younger brother and wanted to do all those things to him which were necessary for his progress and more subtler discoveries in the domain of mathematics by him. Prof. Hardy expressed his views in these words. "There was one great puzzle. What was to be done in the way of teaching him modern mathematics ? The limitations of his knowledge were as startling as its profundity. Here was a man who could work out modular equations and theorems of complex multiplications to orders unheard of, whose mastery of continued fractions was on

the formal side at any rate, beyond that of any mathematician in the world who had found for himself the fractional equation of the zeta function and dominant terms of many of the most famous problems in the analytic theory of numbers, and he had never heard of a doubly periodic function or of Cauchy's theorem and had indeed but the vaguest idea of what a function of a complex variable was. His ideas as to what constituted a mathematical proof were of the most shadowy description. All his results, new or old, right or wrong had been arrived at by a process of mingled argument, intuition and induction of which he was entirely unable to give any coherent account".

On Nov. 11, 1915 Prof. Hardy sent a letter to the Registrar of University of Madras expressing that "Ramanujan was much handicapped by the war, Mr. Littlewood, who would naturally have shared his teaching with me, has been away, and one teacher is not enough for so fertile, a pupil ...... He is beyond question the best Indian mathematician of modern times ..... He will always be rather eccentric in his choice of objects and methods of dealing with them ..... But of his extraordinary gifts there can be no question ; in some ways he is the most remarkable mathmatician I have ever known."

For April 1914 to April 1917, Ramanujan got 21 research papers published. On definite integrals, Riemann's Zeta function partition and combinatorial analysis and number theory, out of which five papers were in joint authorship with Prof. Hardy. In 1915 his paper "Highly Composite Number" consisting of 63 pages and expressing 269 equations was published in the journal of the "London Mathematical Society". For this extraordinary work he was awarded the Bachelor's degree from Trinity college. Prof. Hardy expressed his views about this paper in these words "Mr. Ramanujan's elaborate memoir on "Highly Composite Numbers" contains an account of the longest and perhaps the most important connected piece of work, which he has done since his arrival in England. A highly composite number is a number which has more divisors than any smaller number, which is so to say, as unlike a prime, as a number can be. Thus 2, 4, 6, 12, 24, 36, 48, 60, 120 and 180 are the first such numbers. Mr. Ramanujan shows how by reasoning of an elementary but highly ingenuous character, we can obtain surprisingly accurate information as to the structure of highly composite numbers". In this paper Ramanujan also gave largest highly composite number as

$$6746328388800 = 2^6, 3^4, 5^2, 7^2, 11, 13, 17, 19, 23.$$

Prof. Hardy in the above mentioned report further said tha "India has produced many talented mathematicians in recent years a number of whom have come to Cambridge and attained high academic distinction. They will be the first to recognise that Mr. Ramanujan's work is of a different category. In him, India now possesses a pure mathematician of first order, whose achievements suggest the brightest hopes for its scientific future."

The mathematical researches of Ramanujan brought him high praise and because of such amazing contributions in the field of Pure Mathematics, he was elected a Fellow of Royal Society on February 28, 1918 being a "research student in mathematics distinguished as a pure mathematician particularly for his investigations in elliptic functions and the theory of numbers" and thereafter, on October 13, 1918 he was also elected a Fellow of the Trinity College. With this fellowship he became entitled to get annual stipend of 250 pounds for six years. On the election of Ramanujan to the Fellowship of Trinity College Prof. Hardy wrote to the Registrar of University of Madras that "He (Ramanujan) will return to India with a scientific standing and reputaion such as no Indian has enjoyed before, and I am confident that India will regard him as the treasure he is. His natural simplicity and modesty has never been affected in the least by success—indeed all that is wanted is to get him to realise that he really is a success".

On Ramanujan's exemplary works university of Madras also decided to grant him a yearly allowance of 250 pounds for five years from April 1, 1919 and also offered a lump-sum amount to meet other allied expenditures. When Ramanujan got information about this scholarship, he wrote to the Registrar of university of Madras that ".... I greatfully accept the very generous help which the university offers to me. I feel, however that after my return to India, which I expect to happen as soon as arrangements can be made, the total amount of money to which I shall be entitled will be much more than I shall require. I should hope that after my expenses in England have been paid, sixty pounds a year will be paid to my parents and that the surplus, after my necessary expenses are met, should be used for some educational purposes, such in particular as the reduction of school fees for poor boys and orphans and provision of books in schools. No doubt it will be possible to make an arrangement about this after my return. I feel very sorry that, as I have not been well, I have not been able to do as much mathematics during the last two years as before. I hope that I shall

soon be able to do more and certainly do my best to deserve the help that has been given me".

Ramanujan fell seriously ill in 1917 and was admitted to a nursing home at Cambridge and thereafter shifted to other hospitals at Wells, at Matlock and in London. He was sufferring from pulmanory tuberclosis, which developed probably due to unfavourable climate, nonavailability of proper nourishment being vegitarian and cooking food himself in addition to strenuous and highly engaged acadmic works. When he was at Putney Nursing Home, Prof. Hardy went there and made the conversations

"How are you": Hardy

"Better, thank you" : Ramanujan

Prof. Hardy was very sad on seeing the detoriating health condition of Ramanujan and said "the number of the taxicab in which I came here was 1729. The number seems to be a dull one and I hope it is not an unfavourable omen"

Ramanujan immediately replied with shining eyes :

"No, it is very interesting number, in fact, it is the smallest number expressible as a sum of two cubes in two different ways"

$$(1729 = 12^3 + 1^3 = 10^3 + 9^3)$$

Prof. Hardy was amazed by the answer and said : "Do you know the answer to the corresponding problem for the fourth power" ?

Ramanujan said "I can see no obvious examples. But I think that the first such number must be very large".

He was right in his contemplation, as this number was already discovered by another famous mathematician Euler of eighteenth century.

$$635318657 = 59^4 + 158^4 = 133^4 + 134^4$$

This conversation fully substantiate the view of Prof. Littlewood about Ramanujan that "every positive integer was one of Ramanujan's personal friends".

With the progression of time, there was no improvement in the health condition of Ramanujan so his well wishers in England decided to send him back to India, with a hope that his health condition may improve in a congenial climate and a homely atmosphere. Finally, on Feb., 27, 1919, Ramanujan left England by ship S.S. Nagoya and after the journey of four weeks he reached Bombay and thereafter Madras on April 2, 1919. Indian Mathematical Society convened a meeting on April 1, 1919 and passed a resolution in his appreciation as "Mr. S. Ramanujan, B.A.

F.R.S., this distinguished mathematician has returned to Madras in somewhat different health after a prolonged stay at Cambridge. By this unique mathematical talents and by the amount of useful and original work, he has raised India in the estimation of the outside world. We extend our most cordial welcome to him and most fervently pray that he may be soon restored to his full vigour, to prosecute his glorious work in the scientific world".

From April to June he was in Madras and then went to his maternal grandfather's house in Erode, where he stayed about eight weeks and finally left for his native place Kumbakonam in the month of September. Again, in January 1920, he returned to Madras, where he was given best possible treatment. On January 12, 1920 from Madras Ramanujan wrote a letter to Prof. Hardy that "I am extremely sorry for not writing you a single letter upto now ..... I discovered very interesting functions recently which I call Mock-Theta Functions, unlike the False-Theta Functions (studied partially by Prof. Rogers in his interesting paper), they enter into mathematics as beautifully as the ordinary functions. I am sending you with this letter some examples".

Regarding the discovery of this Mock-Theta-Function, Prof. Watson said in the valadictory address of London Mathematical Society on November 14, 1936 that "Ramanujan's discovery of the Mock-Theta Functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the Mock-Theta Functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that garden of Proserpine where

"Pale, beyond porch and portal
Crowned with calm leaves, she stands
who gathers all things mortal
with cold immortal hands".

Struggling with dreaded disease, he finally succumbed to it on April 26, 1920 at the age of 32 years, four months and four days. It was a great loss to the world of mathematics, whose brightest star disappeared even during its most shining period. Prof. E.T. Bell in his book "Men and Mathematics" opined that "even expert analysts hail Ramanujan as a gift from heaven, his all but supernatural insight into apparently unrelated formulas reveals hidden trails leading from one territory to another and analysts have new tasks

provided for them in clearing the trails". Ramanujan, in fact was an ardent seeker of mathematical knowledge. Whatever problems he faced throughout his life in different forms could not deter him to march ahead in his journey towards higher pleateus of mathematics. The observations of Richard Asky are quite appropriate in this regard that "try to imagine the quality of Ramanujan's mind, one which drove him to work unceasingly while deathly ill and one great enough to grow deeper while his body became weaker. I stand in awe of his accomplishments, understanding is beyond me. We would admire any mathematician whose life's work was half of what Ramanujan found in the last year of his life while he was dying".

After his death, the only treasure left with us are his notebooks. His first handwritten notebook consists of 351 pages with almost sixteen chapters, the second notebook consisting 256 pages in twenty one chapters alongwith 100 unorganized pages and third notebook only of thirty three pages. When he left England, his first notebook remained with Prof. Hardy, which he later on edited in the year 1923 covering 47 main theorems and several other related corrolaries. In 1957, the Tata Institute of Fundamental Research published two facsimile volumes of these notebooks and finally from 1985 onwards Prof. Berndt got published Ramanujan's notebooks in five volumes from Springer–Verlage. Ramanujan's important results were in the areas of gamma functions, continued fractions, modular forms, divergent series, hypergeometric series, highly composite numbers, partition function and its asymptotics, prime number theory and mock-theta functions.

Some of his interesting results are given below for quick references of the readers :

(1) In his early days, Ramanujan was very much fascinated with the magic square and devised some new rules to construct the magic squares of different orders *e.g.*

| | | |
|---|---|---|
| 15 | 1 | 11 |
| 5 | 9 | 13 |
| 7 | 17 | 3 |

(3 × 3 magic square)

Dat. O.

| | | | |
|---|---|---|---|
| 22 | 12 | 18 | 87 |
| 28 | 59 | 40 | 12 |
| 80 | 3 | 37 | 19 |
| 9 | 65 | 44 | 21 |

(4 × 4 magic square)

(2) **Partition of Natural Numbers.** For any given natural number partition is a process of splitting that number into smaller natural numbers such that their sum remains always equal to given number *e.g.*

for $n = 5$, total partitions are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1

So P(5) *i.e.* total partitions for 5 are 7.

Similarly, for other natural numbers also the value of total partitions can be calculated as, P(8) = 22, P(11) = 56, P(16) = 231, P(20) = 627, P(200) = 3972999029388

Ramanujan also gave conjecture related to these partitions by expressing that

P($5n$ + 4) is divisible by 5

P($7n$ + 5) is divisible by 7

P($11n$ + 6) is divisible by 11

which shows that all natural numbers of the form $5n + 4$, when splitted will give the value of total partitions in a multiple of 5, *i.e.* P(4), P(9), P(14), P(19), P(24) ..... will be divisible by 5 ; P(5), P(12), P(19), P(26) ..... will be divisible by 7 and P(6) , P(17), P(28) ..... will be divisible by 11 .

Ramanujan also established one beautiful identity related to partitions of natural numbers as

$$P(4) + P(9)\,x + P(14)x^2 + P(19)x^3 + \ldots$$
$$= \frac{5[(1-x^5)(1-x^{10})(1-x^{15})(1-x^{20})\ldots]^5}{[(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots]^6}$$

(*iii*) Ramanujan solved the problems related to the nested square root in following way,

$$\sqrt{13+2\sqrt{15+3\sqrt{17+4\sqrt{19+5\sqrt{21+\ldots}}}}}$$

(take a function $f(n) = n(n+4)$)

$$n(n+4) = n\sqrt{n^2+8n+6}$$

$$= n\sqrt{(2n+11)+(n^2+6n+5)}$$

$$= n\sqrt{(2n+11)+(n+1)\sqrt{n^2+10n+25}}$$

$$= n\sqrt{(2n+11)+(n+1)\sqrt{(2n+13)+(n+2)\sqrt{n^2+12n+36}}}$$

$$= n\sqrt{(2n+11)+(n+1)\frac{\sqrt{(2n+13)+(n+2)\sqrt{(2n+15)+(n+3)\sqrt{n^2+14n+49}}}}{\sqrt{(2n+13)+(n+2)\sqrt{(2n+15)+\ldots}}}}$$

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$$= n\sqrt{(2n+11)+(n+1)\sqrt{(2n+13)+(n+2)\sqrt{(2n+15)+(n+3)\sqrt{(2n+17+\ldots}}}}$$

put $n = 1$, and get

$$1(1+4) = 1\sqrt{13+2\sqrt{15+3\sqrt{17+4\sqrt{19+\ldots}}}}$$

or $$\sqrt{13+2\sqrt{15+3\sqrt{17+4\sqrt{19+\ldots}}}} = 5$$

(*iv*) Ramanujan gave several approximations of $\pi$, few of them are given below

$$\frac{7}{3}\left(1+\frac{\sqrt{3}}{5}\right) = 3.14162$$

$$\frac{63}{25}\left(\frac{17+15\sqrt{5}}{7+15\sqrt{5}}\right) = 3.1415926358\ \ldots\ldots$$

$$\left(9^2 + \frac{19^2}{22}\right)^{1/4} = 3.141592652\ .....$$

$$\frac{355}{113}\left(1 - \frac{0.0003}{3533}\right) = 3.1415926535897943.....$$

$$\pi = \frac{12}{\sqrt{310}} \log\left[\frac{1}{4}(3+\sqrt{5})(2+\sqrt{2})\left[(5+2\sqrt{10}+\sqrt{61+20\sqrt{10}}\right]\right]$$

$$\pi = \frac{24}{\sqrt{142}} \log\left[\left(\frac{10+11\sqrt{2}}{4}\right)^{1/2} + \left(\frac{10+7\sqrt{2}}{4}\right)^{1/2}\right]$$

→ Ramanujan's talent – [illegible]

Ramanujan also obtained the expansions of $\frac{1}{\pi}$ in the form of infinite series as

$$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right) + .....$$

$$\frac{16}{\pi} = 5 + \frac{47}{64}\left(\frac{1}{2}\right)^3 + \frac{89}{64^2}\left(\frac{1.3}{2.4}\right)^3 + .....$$

$$\frac{27}{4\pi} = 2 + 17 . \frac{1}{2} . \frac{1}{3} . \frac{2}{3}\left(\frac{2}{27}\right) + 32 . \frac{1.3}{2.4}\ \frac{1.4}{3.6}\ \frac{2.5}{3.6}\left(\frac{2}{27}\right)^2 + .....$$

$$\frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6}\ \frac{1}{2}\ \frac{1.3}{3^2} + \frac{53883}{99^{10}}\ \frac{1.3}{2.4}\ \frac{1.3.5.7}{4^2.8^2} + ......$$

$$\frac{\pi}{2} \log 2 = 1 + \frac{1}{2} . \frac{1}{3^2} + \frac{1.3}{2.4} . \frac{1}{5^2} + .....$$

$$\frac{105}{\pi^4} = \left(1 + \frac{1}{2^4}\right)\left(1 + \frac{1}{3^4}\right)\left(1 + \frac{1}{5^4}\right) + .....$$

Ramanujan also gave the proof for the famous geometrical problem related to squaring of a circle. Ramanujan's results have numerous applications in different branches of sciences. His approximations of $\pi$ are providing the basis for the the festest computer algorithm to determine the value of $\pi$ to several places of decimals by the supercomputers. His partition theory is being applied to a new branch of science related to Super String Theory. His formulae are also applied in the field of crystallography. Elaborating these applications Prof. George Andrews stated that

"His (Ramanujan's) mathematics is being applied more widely. It application in theoretical physics is rising. The rise of compute algebra makes it interesting to study somebody who seems like h had a computer algebra package in his head".

To honour S. Ramanujan, Government of India issued a commemorative stamp on his 75th birth day in 1962. Ramanujm's birth day *i.e.* December 22 is also celebrated every year as state IT day in Tamilnadu. International centre for Theoretical Physics in association with the I.M.U. also instituted a prize in the name of S Ramanujan for young mathematician from developing countries Considering remarkable achievements of Srinivas Ramanujan in the field of mathematics praise worthy. Jawhar Lal Nehru also wrote in his book "Discovery of India" that "Mathematics in India inevitably makes one think of one extraordinary figure of recent times. This was Srinivasa Ramanujan. Born in a poor Brahmin family in south India, having no opportunities for a proper educations, he became a clerk in the Madras Port Trust. But he was bubbling over with some irrepressible quality of instinctive genius and played about with numbers and equations in his spare time. By a lucky chance he attracted the attention of a mathematician who sent some of his amateur work to Cambridge in England. People there were impressed and a scholarship was arranged for him. So he left his clerk's job and went to Cambridge and during a very brief period there did work of profound value and amazing originality. The Royal Society of England went rather out of their way and made him a Fellow, but he died two years later, probably to tuberculosis, at the age of thirty three. Professor Julian Huxlay has, I believe, referred to him somewhere as the greatest mathematician of the century".

Finally, it will be appropriate to quote James R. Newman that "India has from time to time posssessed mathematicians of great power. They may be traced through the ages back to the later Greek period. But judged by absolute standards of greatness, among all mathematicians of the East, the genius of Ramanujan appears to be supreme".

They all told in equal voice that

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